Finance

DRM-Finance

N° 2008-03

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March 26, 2008

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*The financial support of the GIP-ANR ("Croyances" project) and of the Risk Foundation (Groupama chair) is gratefully acknowledged.

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Abstract

We address the problem of a social planner who gathers data on experts' consumption discount rates and wants to infer the socially efficient consumption discount rate. We adopt an equilibrium approach with logarithmic utility functions. The equilibrium discount rate is then a weighted average of the individual rates; more impatient experts are more heavily weighted. It decreases with time and converges to the lowest individual discount rate (which is not necessarily the discount rate of the most patient expert). When distributions of tastes and beliefs are independent, our rate is higher than the discount rate proposed by Weitzman (2001).

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1 Introduction

The appropriate social discount rate to apply in public sector cost-benefit analysis is a contentious issue. This is especially true for long term projects, for which financial markets cannot provide any guideline.

As underlined by e.g. Nordhaus (2007) or Weitzman (2007), there is an important distinction between the *utility* social discount rate and the *consumption* social discount rate. The former refers to a pure time preference rate that discounts utility. It reflects the level of impatience or, for long time horizon projects, the relative weights of different people or generations. The latter is the rate used to discount future consumption. There are essentially three determinants of the level of this discount rate. The first determinant is related to a psychological "preference for the present" effect and is represented by the utility discount rate. The more impatient the individuals, the higher the value of one unit of consumption today relative to one unit of consumption tomorrow, the higher the discount rate. But there are other reasons to discount future consumption. The second determinant is related to a wealth effect. Individuals expect that the quantity of available consumption will increase over time. Given decreasing marginal utility of consumption, one unit of consumption today is preferred to one unit of consumption tomorrow. The third determinant is related to a precautionary savings effect. The growth of the quantity of available consumption is uncertain, and if individuals are prudent, this uncertainty should induce them to value more one unit of consumption tomorrow and should reduce the discount rate.

The (extended) Ramsey equation[1] illustrates the distinction and the relation between the

[1] The original Ramsey equation (Ramsey, 1928) was derived in a deterministic setting ($\sigma = 0$) and is given by $R = \rho + \eta\mu$. The extended Ramsey equation corresponds to a direct generalization in a stochastic

utility discount rate and the consumption discount rate. Letting R denote the consumption discount rate, and ρ the utility discount rate, Ramsey formula gives the relation $R = \rho + \eta\mu - \frac{1}{2}\eta(1+\eta)\sigma^2$, where μ is the growth rate of the economy and η is the elasticity of marginal utility. Apart in the specific settings of a stationary economy ($\mu = \sigma = 0$) or a risk neutral investor ($\eta = 0$) or when the wealth and precautionary savings effect cancel out ($\mu = \frac{1}{2}(1+\eta)\sigma^2$), the two discount rates differ. In this paper, we are interested in the properties of the consumption discount rate, since our aim is to determine the value today (in present dollars) of future dollar amounts in order to apply it for cost-benefit analysis.

More precisely, we address the problem of a social planner, who gathers data on individual discount rates (or experts discount rates) and wants to infer the socially efficient consumption discount rate by adopting an equilibrium approach. We emphasize that as in Weitzman (2001), the individual data are about individual *consumption* discount rates[2].

Weitzman (1998, 2001) deal with this problem by adopting a certainty equivalent approach. In this certainty equivalent approach, the social discount factor is given by the probability weighted average of discount functions of the members of the panel. Weitzman (1998) obtains then that the certainty equivalent discount rate is decreasing, and converges to the lowest discount rate. Moreover, Weitzman (2001) starting from a survey of economists estimates the distribution of the individual discount rates, infers the explicit expression of the certainty equivalent rate and proposes "gamma discounting". Gollier (2004) underlines that the approach of Weitzman (1998, 2001) amounts to ranking the projects according to

setting. For the sake of completeness, we rederive it in the Appendix.

[2] As Weitzman (2001) makes it clear in his questionnaire : "What I am here after is the relevant interest rate for discounting real-dollar changes in future goods and services –as opposed to the rate of pure time preference on utility". Moreover, the fact that the given rates are on average equal to 4% confirms that the experts actually gave their discount rate for consumption.

their expected net present value. By adopting the criterion that projects should be ranked according to their expected future value, Gollier (2004) reaches opposite conclusions and concludes that "both criteria are arbitrary as they do not rely on realistic preferences of human beings towards risk and time" suggesting that an equilibrium analysis is maybe the cost to be paid to make policy recommendations that have an economic sense.

Our approach to aggregate experts discount rates into a consensus discount rate is the following. We consider that each expert in the panel has consulted an equilibrium model, calibrating it with her own tastes and beliefs parameters, in order to propose her individual discount rate. For instance, the expert applies the Ramsey formula and gives an individual discount rate R^i that corresponds to her own pure time preference rate (or as previously underlined, to her own conception of intergenerational equity) and her own belief about the future growth of the economy. The divergence in the proposed individual discount rates stems then from divergence in individual tastes and beliefs. We propose to consider a complete markets model with heterogeneous logarithmic utility agents endowed with the beliefs and tastes of the experts and to adopt the equilibrium discount rate in this model as the consensus rate. We show that the equilibrium consumption discount rate is a weighted average of the individual discount rates. It coincides with Weitzman (1998, 2001) certainty equivalent discount rate when all experts have the same pure time preference rate. In a more general setting, the discount rates of the more impatient experts are granted a higher weight in the average. When tastes and beliefs are independent, the equilibrium discount rate is higher than the certainty equivalent discount rate for all horizons. Furthermore, the equilibrium discount rate has the following properties: it decreases with time and it converges to the lowest individual discount rate in the economy. This lowest discount rate corresponds to the discount rate of the most patient expert when all experts share the same belief, to the discount rate of the most pessimistic expert when all experts have the same impatience rate and the same level of confidence or to the discount rate to the least confident expert when all experts have the same impatience rate and the same level of pessimism. These properties hold for both constant and decreasing pure time preference rates. We also determine which concepts of stochastic dominance on the individual characteristics lead to a clear impact on the equilibrium discount rate. Roughly speaking, more pessimism, more patience, more doubt as well as more heterogeneity in individual discount rates reduce the equilibrium discount rate. Finally, a numerical example is constructed starting from Weitzman (2001)'s data. This example suggests using the following approximation of within-period marginal discount rates for long term public projects: Immediate Future about 5 per cent; Near Future about 4 percent; Medium Future about 3 percent; Distant Future about 1.5 per cent and Far-Distant Future about 0 per cent. Except for the Far-Distant Future, these rates are (slightly) higher than those obtained by Weitzman (2001) due to the fact that more impatient experts are more heavily weighted at the equilibrium.

Note that our results also permit to derive the equilibrium utility discount rate (pure time discount rate). It suffices to consider the case where there is no beliefs heterogeneity. The aggregation of individual utility discount rates has been studied by, among others, Reinschmidt (2002) through a certainty equivalent approach, Gollier-Zeckhauser (2005) and Nocetti (2008) through a Benthamite approach, and Lengwiler (2005) through an equilibrium approach. We obtain that the equilibrium utility discount rate is a weighted average of the individual ones. Our formulas are analogous to those of Lengwiler (2005) and coincide with those of Nocetti (2008) and Gollier-Zeckhauser (2005) for specific choices of Pareto weights.

We emphasize that, while these papers aim at aggregating individual *utility* discount rates, the aim of the present paper is to aggregate individual *consumption* discount rates and to do it through an *equilibrium* approach.

All proofs are in the Appendix.

2 Equilibrium discount rate

Let us consider n experts that propose different discount rates (R^i) for cost-benefit analysis of public projects as in Weitzman (2001).

We assume that the discount rate proposed by expert i for costs or benefits occurring at date t comes from a general equilibrium model with log utility and lognormal aggregate consumption at date t. According to the extended Ramsey equation, the consumption discount rate R^i proposed by expert i is given by

$$R^i = \rho_i + \mu_i - \sigma_i^2$$

where $\rho_i > 0$, μ_i and σ_i^2 are respectively the pure time preference rate, the mean and the variance (by unit of time) of the distribution of the growth rate of aggregate consumption that the expert uses in order to calibrate the model. The divergence on the discount rates (R^i) results then from divergence on these parameters.

Assuming that experts differ in their expectation about the growth rate is fairly natural. Indeed, the expected growth rate reflects the opinion about the future. It suffices to look at experts forecasts to realise that there is no consensus about the future of the economy.

Indeed, forecasting for the coming year is already a difficult task. It is natural that forecasts for the next century/millennium are subject to potentially enormous divergence. It is doubtful that agents or economists currently have a complete understanding of the determinants of long term economic evolutions. It is also natural to assume that experts differ in their pure time preference rate since it may reflect their point of view about intergenerational equity as well as one's level of impatience. The important debate among economists (and also among philosophers) on the notion of intergenerational equity is an illustration of this possible divergence. Some will argue that intergenerational choices should be treated as intertemporal individual choices leading to weigh more present welfare. Others will argue that fundamental ethics require intergenerational neutrality and that the only ethical basis for placing less value on the welfare of future generations is the uncertainty about whether or not the world will exist and whether or not these generations will be present.

The problem now is to determine how to aggregate these experts' recommended discount rates into a consensus discount rate. We propose to integrate our experts as agents with heterogeneous tastes and beliefs in a general equilibrium model. More precisely, we shall consider a complete markets economy with heterogeneous agents endowed with the beliefs and tastes chosen by the experts and we shall adopt the equilibrium discount rate in this economy as our consensus discount rate. We suppose that there is no specific reason to discriminate between the experts, hence we assume that our agents have the same initial endowment. In the next, agents with the same tastes and beliefs are grouped together.

To summarise, we have

- N groups of agents,

- $w_i \equiv$relative size of group i,
- $\rho_i \equiv$ pure time preference rate of the agents in group i,
- $t \equiv$ the time at which a cost or benefit is incurred, relative to the present time,
- $\ln\mathcal{N}((\mu_i - \frac{1}{2}\sigma_i^2)t, \sigma_i^2 t) \equiv$ group i's anticipated distribution[3] of aggregate consumption at date t,
- log utility functions,
- $R^i \equiv \rho_i + \mu_i - \sigma_i^2$, the individual discount rate for agents in group i, i.e. the equilibrium discount rate that would prevail if the economy was made of group i agents only.

The weights w_i model then the distribution of experts characteristics. Another way to interpret the model is to take as primitive an economy with different social groups where each expert is representative of a given group ($N = n$). With such an interpretation, w_i corresponds to the relative size of the considered group (in terms of wealth).

We denote by A_t the equilibrium discount factor for horizon t, i.e. the price at date 0 of \$1 at date t. We denote by $R_t \equiv -\frac{1}{t}\ln A_t$ the discount rate for horizon t, i.e. the rate which if applied constantly for all intervening years would yield the discount factor $A_{t.}$ We denote by $r_t \equiv -\frac{A_t'}{A_t}$ the marginal discount rate for horizon t, i.e. the rate of change of the discount factor. We have $R_t = \frac{1}{t}\int_0^t r_s ds$. Marginal and average rates of discount coincide when the discount rate is constant. In particular, for all i, the individual marginal discount rate r^i coincides with the individual discount rate R^i. However, the distinction between the two

[3]This is the case for instance if aggregate consumption is a geometric Brownian motion with drift μ_i and volatility σ_i.

notions of discount rates can become important when the discount rate is time dependent (Groom et al., 2005).

Proposition 1 *1. The equilibrium discount rate is given by*

$$R_t \equiv -\frac{1}{t} \ln \sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} \exp -R^i t. \tag{1}$$

2. The equilibrium marginal discount rate is given by

$$r_t \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp\left(-r^i t\right)}{\sum_{j=1}^{N} w_j \rho_j \exp\left(-r^j t\right)} r^i. \tag{2}$$

3. In the case of homogeneous beliefs ($\mu_i = \mu, \sigma_i = \sigma$), the equilibrium marginal discount rate is given by

$$r_t \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp\left(-\rho_i t\right)}{\sum_{j=1}^{N} w_j \rho_j \exp\left(-\rho_j t\right)} \rho_i + \mu - \sigma^2. \tag{3}$$

As in the certainty equivalent approach of Weitzman (1998), the consensus discount rates obtained through our equilibrium approach are averages of the individual discount rates proposed by the experts. However, except in the case of homogeneous pure time preference rates, i.e. $\rho_i = \rho$ for all i, our expressions for the rates are different from those of Weitzman (1998). Our equilibrium discount rates are weighted averages, the weights being proportional to the pure time preference rates, while the certainty equivalent rates are unweighted averages of the individual discount rates R^i. This means that there is a bias towards the more impatient agents in the consensus equilibrium discount rate. A possible interpretation is as follows. When considering its attitude towards postponing aggregate consumption, the group must take into account the rate of impatience of those members who will have to postpone their consumption, which induces a bias towards the more impatient members of the group[4].

In the case with homogeneous beliefs, Equation (3) involves the covariance between ρ_i and $\exp -\rho_i t$ as in Lengwiler (2005). Equation (3) also gives us the expression for the consensus utility discount rate $\rho \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp(-\rho_i t)}{\sum_{j=1}^{N} w_j \rho_j \exp(-\rho^j t)} \rho_i$. Although of the same nature, it is slightly different from the one obtained through the Benthamite approach of Gollier (2005) or Nocetti (2008). Indeed, our weights in the weighted averages of the ρ_i are given by the quantities $w_i \rho_i \exp -\rho_i t$ whereas they are given by $\lambda_i \exp -\rho_i t$ in Gollier (2005) or Nocetti (2008), where the (λ_i) are Pareto weights chosen by the social planner. Notice that this means that our equilibrium approach and the Benthamite approach would lead to the same social utility discount rate if the Pareto weights were proportional to $w_i \rho_i$.

The following corollary precises the relation between the different formulas for the discount rate that can be found in the literature.

Corollary 2 *1. The equilibrium discount rate is lower than the pure time preference weighted arithmetic average of the individual discount rates, i.e.*

$$R_t \leq \sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} R^i,$$

$$r_t \leq \sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} r^i.$$

[4] Note that pessimism/optimism in the form of a higher μ_i or overconfidence/doubt in the form of a lower σ_i^2 have no impact on the relative weights, but this might be due to the specific myopic utility function.

2. *If the tastes and beliefs characteristics ρ_i and $b_i \equiv \mu_i - \sigma_i^2$ are independent or if they are anticomonotonic, i.e. individuals with higher tastes characteristics ρ_i have lower beliefs characteristics b_i, the equilibrium discount rate is higher than an average of the individual discount rates, i.e.*

$$R_t \geq -\frac{1}{t} \ln \sum_{i=1}^{N} w_i \exp -R^i t.$$

3. *If the tastes and beliefs characteristics ρ_i and $b_i \equiv \mu_i - \sigma_i^2$ are independent, the equilibrium marginal discount rate is higher than an average of the individual marginal discount rates, i.e.*

$$r_t \geq \sum_{i=1}^{N} \frac{w_i \exp -r^i t}{\sum_{j=1}^{N} w_j \exp -r^j t} r^i.$$

As expected, the discount rate to use is lower than the simple arithmetic average (with the same weights) of the individual discount rates. Moreover, when tastes and beliefs characteristics are independent, our discount rates are higher than those of Nocetti (2008) and Weitzman (2001). This is intuitive since our weights are given by the time pure preference rates ρ_i hence higher weights are granted to higher individual discount rates. However, our equilibrium discount rates share with the consensus pure time preference rate of Reinschmidt (2002), Gollier (2005), Nocetti (2008), the certainty equivalent rate of Weitzman (1998, 2001) and the equilibrium homogeneous beliefs rate of Lengwiler (2005) the following properties.

Corollary 3 *The equilibrium discount rates have the following properties*

1. *R_t and r_t decrease with t,*

2. *The asymptotic equilibrium discount rates are given by the lowest individual discount rate, i.e.* $R_\infty = r_\infty = \inf_i r^i = \inf_i R^i$.

This means that the equilibrium approach leads to decreasing discount rates, not only utility discount rates, but also consumption discount rates. This leads to use lower discount rates for long term projects in a cost-benefit analysis. The asymptotic discount rate is given by the lowest individual discount rate (lowest $R^i \equiv \rho_i + \mu_i - \sigma_i^2$). This rate corresponds to the discount rate of the most patient agent (lowest ρ_i) when there is no beliefs heterogeneity, or to the most pessimistic agent (lowest μ_i) when there is no pure time preference rate heterogeneity and all the agents have the same volatility parameter or to the least confident agent (highest σ_i^2) when there is no pure time preference rate heterogeneity and all the agents have the same drift parameter.

3 Specific distributions and dominance properties

Let us now determine the equilibrium discount rate for specific distributions of the individual discount rates (R^i). The problem is that according to Equations (1) and (2), we need to make extra assumptions on the joint distribution of (ρ_i, R^i) in order to determine R.

Consider first the case with homogeneous pure time preference rates $\rho_i = \rho$, and with a normal distribution $\mathcal{N}(m, v^2)$ on the beliefs parameters $b_i = \mu_i - \sigma_i^2$. The discount rates (R^i) then follow a normal distribution $\mathcal{N}(\rho + m, v^2)$ and we easily obtain that $R_t = \rho + m - \frac{v^2}{2}t$. Reinschmidt (2002) obtains a similar formula for the consensus utility discount rate when the individual utility discount rates follow a normal distribution.

Consider now Weitzman (2001)'s data and suppose that utility discount rates and beliefs are independently and gamma distributed. We obtain the following result.

Proposition 4 *If pure time preference rates ρ_i and beliefs $b_i = \mu_i - \sigma_i^2$ are independently distributed*[5] *with $\rho_i \sim \gamma(\alpha_1, \beta_1)$ and $b_i \sim \gamma(\alpha_2, \beta_2)$, then*

1. $R_t = -\frac{\alpha_1+1}{t}\ln\frac{\beta_1}{\beta_1+t} - \frac{\alpha_2}{t}\ln\frac{\beta_2}{\beta_2+t}$ *and* $r_t = \frac{\alpha_1+1}{\beta_1+t} + \frac{\alpha_2}{\beta_2+t} = \frac{m_1^2+v_1^2}{m_1+tv_1^2} + \frac{m_2^2}{m_2+tv_2}$ *where* (m_1, v_1^2) *and* (m_2, v_2^2) *respectively denote the mean and variance of* (ρ_i) *and* (b_i).

2. *If* $\beta_1 = \beta_2$ *then* $R^i \sim \gamma(\alpha, \beta)$ *with* $\alpha = \alpha_1 + \alpha_2$, $R_t = R_t^W + \frac{1}{t}\ln\left(1 + \frac{t}{\beta}\right)$ *and* $r_t = \frac{m^2+v^2}{m+tv^2} = r_t^W + \frac{1}{\beta+t}$ *where* r_t^W *and* R_t^W *respectively denote the marginal discount rate and the discount rate obtained through the certainty equivalent approach of Weitzman and where* (m, v) *denote the mean and variance of* (R^i).

A decrease in the mean m_2 or an increase in the variance v_2^2 of the individual beliefs (b_i) decreases the marginal discount rate r_t (hence the discount rate R_t). The same result occurs with a decrease in the mean m_1 of the individual pure time preference rates (ρ_i). An increase in the variance v_1^2 of the individual pure time preference rates (ρ_i) decreases the marginal discount rate r_t for t large enough.

When beliefs and tastes are independent and follow gamma distributions with the same parameter β, the distribution of the individual discount rates R^i or r^i is a sufficient statistics for the equilibrium discount rate. A decrease in the mean m of the individual discount rates (R^i) decreases the marginal discount rate and an increase in the variance v^2 of the individual discount rates (R^i) decreases the marginal discount rate r_t for t large enough.

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[5] *Recall that the density function of a gamma distribution $\gamma(\alpha, \beta)$ is given by $\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp(-\beta x)$. Its mean m and its variance v^2 are respectively given by $m = \frac{\alpha}{\beta}$ and $\frac{\alpha}{\beta^2}$.*

In order to calibrate this model with two independent gamma distributions on Weitzman (2001)'s data, we determine $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ such that $m_1 + m_2 = \bar{m}$ and $v_1^2 + v_2^2 = \bar{v}^2$ where $\bar{m}$ and $\bar{v}^2$ respectively denote the mean and the variance of the individual discount rates computed on Weitzman(2001)'s sample. We further impose that $\frac{m_1}{v_1} = \frac{m_2}{v_2}$ (same ratio between mean and standard deviation for both distributions) which leads to $\alpha_1 = \alpha_2$ and $\frac{\beta_1}{\beta_2} = \frac{m_2}{m_1} = \lambda$ where $\lambda > 0$ is a given parameter. The case $(\alpha_1, \beta_1, \alpha_2, \beta_2) = \left(\frac{m^2}{2v^2}, \frac{m}{v^2}, \frac{m^2}{2v^2}, \frac{m}{v^2}\right)$ corresponds to Weitzman(2001)'s calibration ($\lambda = 1$). We have then a family of stastical models that contains Weitzman (2001)'s statistical model and we maximize the log-likelihood with respect to the parameter λ to choose the best calibration. We obtain $\lambda = 0.4116$ hence $(\alpha_1, \beta_1, \alpha_2, \beta_2) = (1.043, 89.454, 1.043, 36.819)$ and $(m_1, v_1^2, m_2, v_2^2) = (1.16 \times 10^{-2}, 1.30 \times 10^{-4}, 2.83 \times 10^{-2}, 7.69 \times 10^{-4})$. To summarise, the best calibration corresponds to a gamma distribution on the individual pure time preference rates with an average rate among experts equal to 1.16% and a median equal to 0.67% and a gamma distribution on the individual beliefs with an average belief parameter equal to 2.83% and a median equal to 2%. The belief parameter $b = \mu - \sigma^2$ can be interpreted as a risk adjusted growth rate. The values we obtain are then reasonable values for both an average pure time preference rate and an average risk-adjusted growth rate. Stern report considers values for the impatience rate (utility discount rate) between 0.1 and 1.5 and values for the growth rate ranging from 0 per cent to 6 per cent. Arrow (1995) states that the pure time preference rate should be about 1%. Surveying the evidence, the HM Treasury's Green Book (2003) suggests a long run growth rate of 2.1 per cent.

Figure 1 represents the log-likelihood as a function of λ. Figure 2 represents the distribution of the individual discount rates for the parameter λ that maximizes the log-likelihood

($\lambda = 0.4116$) as well as the empirical distribution and Weitzman (2001)'s distribution. Figure 3 represents the corresponding marginal discount rate curve and compares it to the discount rate curve of Weitzman (2001). Table 1 presents the corresponding recommended sliding-scale discount rates.

Proposition 5 *1. If all the agents have the same ρ_i, then a FSD (resp. SSD) shift in the distribution of (R^i) increases the discount rate R_t for all horizons.*

2. If all the agents have the same ρ_i, then a MLR[6] shift in the distribution of the (r^i) increases the marginal discount rate r_t for all horizons.

3. If all the agents have the same beliefs, then a MLR shift in the distribution of the (R^i) increases the discount rate R_t for all horizons.

Roughly speaking, a country where experts are more pessimistic and/or exhibit more doubt about future growth and/or have lower pure time preference rates (more patient or more altruistic with respect to future generations) should apply a lower discount rate for cost-benefit analysis. More heterogeneity in experts beliefs about future growth rates also leads to lower discount rates.

More precisely, suppose that in one population, say (A), we have three equally large groups with discount rates of 2%, 3% and 4%. In a second population (B), there are also three groups with the same anticipated growth rates but their proportion in the population is

[6] *Monotone likelihood ratio dominance (MLR) has been studied by Landsberger and Meilijson (1990) and is defined as follows: a random variable Y dominates a random variable X, if X and Y have densities with respect to some dominating measure ν such that*

$$f_X(x) f_Y(y) \leq f_X(y) f_Y(x) \qquad \text{for all } y \leq x$$

(roughly speaking, the ratio $\frac{f_Y}{f_X}$ is nondecreasing).

$\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$. Population (B) is more pessimistic than population (A) (in the sense of the FSD) and the discount rate to apply is lower for (B). In a third population (C), there are three groups with anticipated growth rates 1%, 3% and 5% and their proportion in the population is $\frac{1}{10}$, $\frac{8}{10}$ and $\frac{1}{10}$. Populations (A) and (C) have the same average level of pessimism but population (C) is more heterogeneous (in the sense of the SSD) than population (A) and the discount rate to apply is lower for (C). Let us assume now that experts provide forecasts with a 95% confidence interval. Let us assume that these intervals in population (A) are given by $[1.5, 2.5]$, $[2.5, 3.5]$ and $[3.5, 4.5]$ while in a fourth population (D) also with three equally large groups, these intervals are given by $[1; 3]$, $[2; 4]$ and $[3; 4]$. There is more doubt in population (D) and the discount rate to apply is then lower for (D). The MLR (monotone likelihood ratio) dominance is stronger than the FSD dominance. Let us consider two populations (E) and (F). In population (E), there are three equally large groups of experts with pure time preference rates respectively equal to 0.5%, 1% and 1.5%. In population (F) there are also three groups with the same pure time preference rates but with proportions in the population respectively equal to w_1, w_2 and w_3. The population (E) is more patient (in the sense of the MLR) if $w_3 < w_2 < w_1$. In this case, the discount rate to apply for cost-benefit analysis is lower for population (E).

4 Extensions and remarks

In this section, we examine essentially three possible extensions: more general subjective and objective distributions for aggregate consumption, time dependent pure time preference rates, and more general utility functions.

Proposition 1 remains valid in a very general Arrow-Debreu setting. Time can be continuous or discrete. We allow for a finite number or a continuum of agents. For this purpose, the set of agents is represented by a measured space $([0,1], \nu)$. Furthermore we do not need to assume specific individual distributions for aggregate consumption. It suffices to assume that agent i has a probability measure Q_t^i that represents the distribution of date$-t$ aggregate consumption from agent i point of view. As in previous sections, agent i has a pure time preference rate ρ_i, a share of total wealth w_i and a log-utility.

Proposition 6 *Let us consider a model with a measured space* $([0,1], \nu)$ *of log-utility agents that have pure time preference rates* (ρ_i), *wealth shares* (w_i) *and date*$-t$ *probability measures* Q_t^i. *We assume that all these probabilities are equivalent, i.e. they agree on the events of zero probability. The equilibrium discount rate is then given by*

$$R_t \equiv -\frac{1}{t} \ln \int \frac{w_i \rho_i}{\int w_j \rho_j d\nu(j)} \exp\left(-R_t^i t\right) d\nu(i) \tag{4}$$

where R_t^i *is the equilibrium discount rate that would prevail if the economy was made of agent* i *only.*

In such a general setting the equilibrium discount rate is still a weighted average of the individual discount rates, the weights being proportional to $w_i \rho_i$.

It is also easy to adapt our approach to the case with time-dependent pure time preference rates $(\rho_i(t))$. We then have $R_t^i = \frac{1}{t}\int_0^t \rho_i(s)ds + \mu_i - \sigma_i^2$ and $r_t^i = \rho_i(t) + \mu_i - \sigma_i^2$.

Proposition 7 *If agents have time-dependent positive pure time preference rates* $(\rho_i(t))$, *wealth shares* (w_i) *and date*$-t$ *distributions for aggregate wealth* $\ln \mathcal{N}((\mu_i - \frac{1}{2}\sigma_i^2)t, \sigma_i^2 t)$, *the* *equilibrium discount rates are given by*

$$R_t \equiv -\frac{1}{t}\ln\sum_{i=1}^{N}\frac{w_i\overline{\rho}_i}{\sum_{i=1}^{N}w_j\overline{\rho}_j}\exp\left(-R_t^i t\right) \tag{5}$$

$$r_t \equiv \sum_{i=1}^{N}\frac{w_i\overline{\rho}_i\exp\left(-\int_0^t r_s^i ds\right)}{\sum_{i=1}^{N}w_j\overline{\rho}_j\exp\left(-\int_0^t r_s^i ds\right)}r_t^i \tag{6}$$

for $R_t^i = \frac{1}{t}\int_0^t \rho_i(s)ds + \mu_i - \sigma_i^2$, $r_t^i = \rho_i(t) + \mu_i - \sigma_i^2$ *and* $\overline{\rho}_i = \left(\int_0^\infty \exp - \int_0^t \rho_i(s)\, dsdt\right)^{-1}$.

If the pure time preference rates $(\rho_i(t))$ *are decreasing with* t*, then the discount rates* R_t *and* r_t *are also decreasing with* t *and we have*

$$\lim_{t\to\infty} R_t = \lim_{t\to\infty} r_t = \inf_i \inf_t r_t^i = \inf_i \left(\mu_i - \sigma_i^2 + \lim_{t\to\infty}\rho_i(t)\right).$$

Let us end this note by a remark on the choice of the utility function. Our approach relies on logarithmic utility functions, essentially for analytical tractability[7]. Indeed, as underlined by Rubinstein (1975), "log utility functions are singular in their capacity to cope with heterogeneous beliefs while not imposing unreasonable restrictions on tastes". This choice unables us to obtain simple formula, while considering reasonable levels of risk aversion. The case with more general power utility functions would be much more difficult to handle. As underlined by Shefrin (2005) and Jouini-Napp (2007), in a power utility functions framework the state-price density is nomore an arithmetic average of the individual state-price densities. It would be then difficult to obtain the equilibrium discount rate as a function of the individual discount rates. Moreover, Jouini-Napp (2007) shows that the log-utility

[7] This choice of η=1 is also made in the Stern Review.

setting is central in the analysis of beliefs heterogeneity: some biases are induced when we deal with power utility function $\frac{c^{1-\eta}}{1-\eta}$ with $\eta \neq 1$, these biases being in opposite directions depending on the position of η with respect to 1. This is then an additional argument in favor of the log-utility setting.

5 Conclusion

In this paper we start with the recognition that divergence among experts on what the discount rate should be is rooted in fundamental differences of opinion about inter-generational equity as well as about future growth of the quantity of available consumption. We propose an equilibrium approach to aggregate the individual discount rates (proposed by experts) into a consensus discount rate. We emphasize that our approach enables to deal with individual consumption discount rates and not only with utility discount rates (pure time preference rates). We obtain that the equilibrium discount rate is a weighted average of the individual discount rates: more impatient experts are more heavily weighted. The equilibrium discount rate is decreasing and converges to the lowest individual discount rate which does not necessarily correspond to the discount rate of the more patient agent. More divergence of opinion among experts leads to lower discount rates for all horizons. More doubt in experts' forecasts (larger confidence intervals) also leads to lower discount rates. Starting from Weitzman (2001)'s data, we show that the very wide spread of opinion on discount rates makes the effective equilibrium rate decline significantly over time from 5 per cent per annum for Immediate Future to 0 per cent per annum for Far-Distant Future.

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Appendix

Derivation of the extended Ramsey equation

The present aggregate consumption is equal to 1 and the date$-t$ aggregate consumption e_t follows a lognormal distribution with parameters $\left(\mu - \frac{1}{2}\sigma^2\right) t$ and $\sigma^2 t$. For instance, such a distribution might result from a geometric Brownian motion with drift μ and volatility σ. The representative agent has a power utility function $u'(c) = c^{-\eta}$ and a pure time preference rate ρ. At the equilibrium, the date t state price density q_t is given by

$$q_t = \exp(-\rho t) u'(e_t)$$

and the discount rate is given by

$$R = -\frac{1}{t} \ln E\left[q_t\right] = \rho - \frac{1}{t} \ln E\left[e^{-\eta}\right].$$

The random variable $e^{-\eta}$ follows a log normal distribution with parameters $-\eta\left(\mu - \frac{1}{2}\sigma^2\right) t$ and $\sigma^2\eta^2 t$. We then have $\ln E\left[c^{-\frac{1}{\eta}}\right] = -t(\eta\mu - \frac{1}{2}\eta\left(1+\eta\right)\sigma^2)$ and

$$R = \rho + \eta\mu - \frac{1}{2}\eta\left(1+\eta\right)\sigma^2.$$

■

Proof of Proposition 1

As proved in a more general setting in Proposition 6 below, we have $A_t = \sum_{i=1}^{N} \gamma_i A_t^i$, with $\gamma_i = \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j}$, from which we easily deduce Equations (1) and (2). ■

Proof of Corollary 2

1. We have

$$R_t \equiv -\frac{1}{t}\ln A_t$$

where A_t is an arithmetic average of the $\exp -R^i t$. Since the arithmetic average is larger than the geometric average, we have

$$\sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} \exp\left(-R^i t\right) \geq \exp - \sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} R^i t.$$

Hence,

$$R_t \leq \sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} R^i.$$

We have

$$\begin{aligned} r_t &\equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp\left(-r^i t\right)}{\sum_{j=1}^{N} w_j \rho_j \exp\left(-r^j t\right)} r^i \\ &= \frac{E^{P_\rho}\left[\exp\left(-rt\right) r\right]}{E^{P_\rho}\left[\exp\left(-rt\right)\right]} \end{aligned}$$

where P_ρ has weights $\frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j}$. Since $\exp\left(-r^i t\right)$ decreases with r^i, we have $E^{P_\rho}\left[\exp\left(-rt\right) r^i\right] \leq E^{P_\rho}\left[r\right] E^{P_\rho}\left[\exp\left(-rt\right)\right]$. Hence

$$r_t \leq E^{P_\rho}\left[r\right] = \sum_{i=1}^{N} \frac{w_i \rho_i}{\sum_{j=1}^{N} w_j \rho_j} r^i.$$

2. Let us denote by P_w the probability measure with weights w_i. Since ρ_i and $b_i \equiv \mu_i - \sigma_i^2$ are independent, we have

$$\begin{aligned}\exp\left(-R_t t\right) &= \frac{E^{P_w}\left[\rho\exp\left(-\rho t\right)\exp-\left(b\right)t\right]}{E^{P_w}\left[\rho\right]}, \\ &= \frac{E^{P_w}\left[\rho\exp\left(-\rho t\right)\right]E^{P_w}\left[\exp -bt\right]}{E^{P_w}\left[\rho\right]}.\end{aligned}$$

Now, since ρ_i and $\exp\left(-\rho_i t\right)$ are anticomonotonic, we have

$$E^{P_w}\left[\rho\exp\left(-\rho t\right)\right] \leq E^{P_w}\left[\rho\right]E^{P_w}\left[\exp\left(-\rho t\right)\right],$$

which gives

$$\begin{aligned}\exp -R_t t &\geq E^{P_w}\left[\exp\left(-\rho t\right)\right]E^{P_w}\left[\exp\left(-bt\right)\right], \\ &\geq E^{P_w}\left[\exp\left(-rt\right)\right].\end{aligned}$$

3. We have

$$r_t = \frac{E^{P_{\exp}}\left[\rho^2\right] + E^{P_{\exp}}\left[\rho b\right]}{E^{P_{\exp}}\left[\rho\right]}.$$

where $P_{\exp}$ denotes the probability measure whose weights are proportional to $w_i\exp(-r^i t)$.

We have then

$$
\begin{aligned}
r_t &\geq \frac{E^{P_{\exp}}[\rho]^2 + E^{P_{\exp}}[\rho b]}{E^{P_{\exp}}[\rho]}, \\
&\geq E^{P_w}[\rho \exp(-\rho t)] \frac{E^{P_w}[\rho \exp(-\rho t)] E^{P_w}[\exp(-bt)] + E^{P_w}[\exp(-bt)\, b] E^{P_w}[\exp(-\rho t)]}{E^{P_w}[\exp(-\rho t)] E^{P_w}[\rho \exp(-rt)]}, \\
&\geq \frac{E^{P_w}[(\rho + b)\exp(-(\rho + b)\, t)]}{E^{P_w}[\exp(-\rho t)] E^{P_w}[\exp(-bt)]}, \\
&\geq \sum_{i=1}^{N} \frac{w_i \exp(-r^i t)}{\sum_{j=1}^{N} w_j \exp(-r^j t)} r^i.
\end{aligned}
$$

■

Proof of Corollary 3 We have $r_t \equiv \sum_{i=1}^{N} \frac{w_i \rho_i \exp\left(-r^i t\right)}{\sum_{j=1}^{N} w_j \rho_j \exp(-r^j t)} r^i$, then $\frac{dr_t}{dt} = \frac{\left(\sum_{i=1}^{N} w_i \rho_i \exp\left(-r^i t\right) r^i\right)^2}{\left(\sum_{i=1}^{N} w_i \rho_i \exp(-r^i t)\right)^2} - \sum_{i=1}^{N} \frac{w_i \rho_i \exp\left(-r^i t\right)}{\sum_{j=1}^{N} w_j \rho_j \exp(-r^j t)} \left(r^i\right)^2$. Let us consider $P_{\rho \exp}$ the probability measure whose weights are proportional to by $w_i \rho_i \exp(-r^i t)$. We have $\frac{dr_t}{dt} = E^{P_{\rho \exp}}[r]^2 - E^{P_{\rho \exp}}[r^2] \leq 0$. The marginal discount rate decreases then with t. Since $R_t = \frac{1}{t} \int_0^t r_s ds$, it follows that the discount rate is also decreasing with t.

Let $r^{i^*} \equiv \inf_i r^i$ and let j be such that $r^j \neq r^{i^*}$, then the relative weight of r^j in Equation (2) converges to zero and $r_t \to_{t \to \infty} r^{i^*}$. The discount rate inherits the same properties. ■

Proof of Proposition 4

If pure time preference rates ρ_i and beliefs $b_i = \mu_i - \sigma_i^2$ are independent and are distributed as follows $\rho_i \sim \Gamma(\alpha_1, \beta_1)$ and $b_i \sim \Gamma(\alpha_2, \beta_2)$, then

$$
\begin{aligned}
A_t &= \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}\frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\frac{\int_0^\infty\int_0^\infty \rho\exp\left(-\left(\rho+b\right)t\right)\rho^{\alpha_1-1}\exp\left(-\beta_1\rho\right)b^{\alpha_2-1}\exp\left(-\beta_2 b\right)d\rho db}{\frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}\int_0^\infty \rho\rho^{\alpha_1-1}\exp\left(-\beta_1\rho\right)d\rho}, \\
&= \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}\frac{\left(\int_0^\infty \exp\left(-\rho\left(\beta_1+t\right)\right)\rho^{\alpha_1}d\rho\right)\left(\int_0^\infty \exp\left(-b\left(\beta_2+t\right)\right)b^{\alpha_2-1}db\right)}{\int_0^\infty \rho^{\alpha_1}\exp\left(-\beta_1\rho\right)d\rho}, \\
&= \left(\frac{\beta_1}{\beta_1+t}\right)^{1+\alpha_1}\left(\frac{\beta_2}{\beta_2+t}\right)^{\alpha_2}.
\end{aligned}
$$

■

Proof of Proposition 5

1. Let us assume that all the agents have the same ρ_i, we have then

$$R_t \equiv -\frac{1}{t}\ln E^{P_w}\left[\exp -Rt\right]$$

where P_w is defined as in the proof of Corollary 2. For a given t, the function $R \to \exp -Rt$ is decreasing (and convex) and, by definition, a FSD (resp. SSD) shift in the distribution of (R^i) decreases the value of $E^{P_w}\left[\exp -Rt\right]$ and increases R_t.

2. We still assume that all the agents have the same ρ_i, we have then

$$r_t = \frac{E^{P_w}\left[r\exp\left(-rt\right)\right]}{E^{P_w}\left[\exp\left(-rt\right)\right]}.$$

Let us consider P_w^1 and P_w^2, two distributions such that $P_w^2 \succeq_{MLR} P_w^1$. By definition, the density $\phi = \frac{dP_w^2}{dP_w^1}$ is nondecreasing in r (in other words $i \to \phi^i$ and $i \to r^i$ are comonotonic). We have then, $\frac{E^{P_w^2}[r\exp -rt]}{E^{P_w^2}[\exp(-rt)]} = \frac{E^{P_w^1}[\phi r\exp -rt]}{E^{P_w^1}[\phi\exp(-rt)]} = \frac{E^{Q_{\exp}}[\phi r]}{E^{Q_{\exp}}[\phi]}$ where $Q_{\exp}$ is defined by a density with respect to P_w^1 equal (up to a constant) to $\exp(-rt)$. Since ϕ is nondecreasing in r, we have

$$E^{Q_{\exp}}[\phi r] \geq E^{Q_{\exp}}[\phi]\, E^{Q_{\exp}}[r],$$

hence

$$\begin{aligned}\frac{E^{P_w^2}[r \exp -rt]}{E^{P_w^2}[\exp(-rt)]} &\geq E^{Q_{\exp}}[r], \\ &\geq \frac{E^{P_w^1}[r \exp -rt]}{E^{P_w^1}[\exp -rt]}.\end{aligned}$$

3. If we now assume that all the agents have the same belief, we have

$$R_t \equiv -\frac{1}{t} \ln \frac{E^{P_w}[\rho \exp -\rho t]}{E^{P_w}[\rho]}.$$

Let us consider P_w^1 and P_w^2, two distributions such that $P_w^2 \succeq_{MLR} P_w^1$. We have then, $\frac{E^{P_w^2}[\rho \exp -\rho t]}{E^{P_w^2}[\rho]} = \frac{E^{P_w^1}[\phi \rho \exp -\rho t]}{E^{P_w^1}[\phi \rho]} = \frac{E^{Q_\rho}[\phi \exp -\rho t]}{E^{Q_\rho}[\phi]}$ where $\phi = \frac{dP_w^2}{dP_w^1}$ and where Q_ρ is defined by a density with respect to P_w^1 equal (up to a constant) to ρ. Since ϕ is nondecreasing in r and then nonincreasing in $\exp -\rho t$, we have

$$E^{Q_\rho}[\phi \exp -\rho t] \leq E^{Q_\rho}[\phi]\, E^{Q_\rho}[\exp -\rho t],$$

hence

$$\begin{aligned}\frac{E^{P_w^2}[\rho \exp -\rho t]}{E^{P_w^2}[\rho]} &\leq E^{Q_\rho}[\exp -\rho t], \\ &\leq \frac{E^{P_w^1}[\rho \exp -\rho t]}{E^{P_w^1}[\rho]}.\end{aligned}$$

■

Proof of Proposition 6

Let M_t^i denote the density of Q_t^i with respect to a given probability P equivalent to all the probability measures Q_t^i. Let us denote by q_t the date$-t$ state-price density (with respect to P) and by y_t^i the consumption of group i. Each group maximizes its aggregate utility $\int_0^\infty \exp(-\rho_i t) E^{Q_t^i}\left[\frac{1}{y_t^i}\right] dt$ under its budget constraint $\int_0^\infty E^P\left[q_t y_t^i\right] dt \leq w_i \int_0^\infty E^P\left[q_t e_t\right] dt$. This leads to the following Euler condition

$$\frac{1}{\lambda_i}\exp(-\rho_i t) M_t^i \frac{1}{y_t^i} = q_t.$$

We have then

$$\frac{1}{\lambda_i}\exp(-\rho_i t) M_t^i \frac{1}{q_t} = y_t^i$$

and summing all these equations leads to

$$q_t = \sum_{i=1}^{N} \frac{1}{\lambda_i}\exp(-\rho_i t) M_t^i \frac{1}{e_t}.$$

Now in an economy made of group i only, we would have

$$\exp(-\rho_i t) M_t^i \frac{1}{e_t} = q_t^i$$

and

$$r_t^i = -\frac{1}{t}\ln E\left[q_t^i\right].$$

If all prices are expressed in terms of today's consumption units, we have $q_0 = 1$ and $\sum_{i=1}^{N} \frac{1}{\lambda_i} = 1$ which leads to

$$q_t = \sum_{i=1}^{N} \frac{1}{\lambda_i} q_t^i$$

hence

$$A_t = \sum_{i=1}^{N} \frac{1}{\lambda_i} A_t^i.$$

It remains to determine the equilibrium weights $\frac{1}{\lambda_i}$. From the Euler and budget conditions we have

$$\int_0^\infty E^P \left[q_t y_t^i \right] dt = \frac{1}{\lambda_i \rho_i} = w_i \int_0^\infty E^P \left[q_t e_t \right] dt$$

which leads to

$$\frac{1}{\lambda_i} = \frac{\rho_i w_i}{\sum_{j=1}^{N} \rho_j w_j}$$

and $q_t = \sum_{i=1}^{N} \frac{\rho_i w_i}{\sum_{j=1}^{N} \rho_j w_j} q_t^i$. Since $A_t = E\left[q_t\right]$, we have

$$A_t = \sum_{i=1}^{N} \frac{\rho_i w_i}{\sum_{j=1}^{N} \rho_j w_j} A_t^i,$$

hence Equation 4. ■

Proof of Proposition 7 It is easy to see that the formulas in the proof of Proposition 6 above have to be adapted as follows

$$\begin{aligned} q_t &= \frac{1}{\lambda_i} \exp(-\int_0^t \rho_i(s) ds) M_t^i \frac{1}{y_t^i} \\ \int_0^\infty E^P \left[q_t y_t^i \right] dt &= \int_0^\infty \frac{1}{\lambda_i} \exp(-\int_0^t \rho_i(s) ds) dt \end{aligned}$$

The same steps as above lead to

$$A_t = \sum_{i=1}^{N} \frac{\overline{\rho}_i w_i}{\sum_{j=1}^{N} \overline{\rho}_j w_j} A_t^i$$

with $\overline{\rho}_i = \left(\int_0^\infty \exp(-\int_0^t \rho_i(s)ds)dt \right)^{-1}$. ■

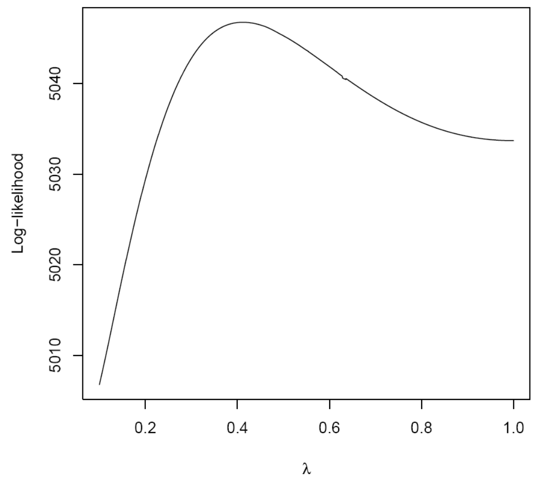

Figure 1: We calibrate a model with two independent gamma distributions (tastes and beliefs) on Weitzman (2001)'s data. We assume that the two distributions are homothetic (the first one is obtained from the second one through a change of variable $x \rightarrow \lambda x$ where λ is a given parameter) and we calibrate the model in order to fit the mean and the variance of the empirical distribution. We have then a family of stastical models that contains Weitzman (2001)'s statistical model (it corresponds to $\lambda = 1$) and we maximize the log-likelihood with respect to the parameter λ to choose the best calibration. We obtain $\lambda = 0.4116$.

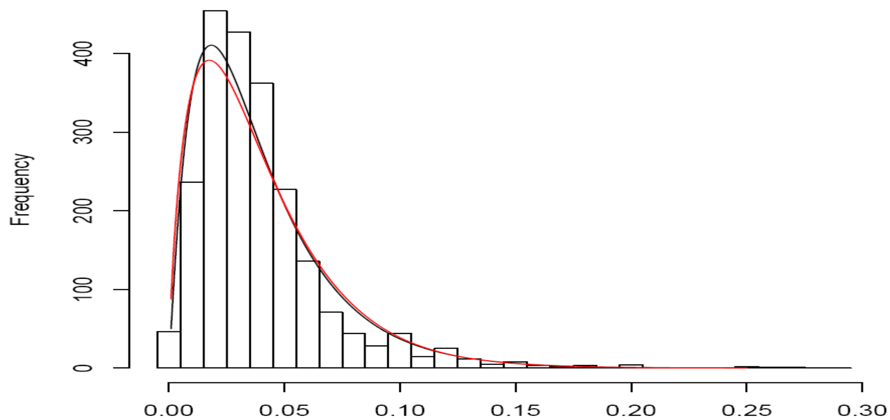

Figure 2: This figure represents the distribution of the individual discount rates for the value $\lambda = 0.4116$ that maximizes the log-likelihood (upper curve) as well as the empirical distribution and Weitzman (2001)'s distribution (lower curve). Our distribution corresponds to the sum of two independent gamma distributions with parameters (α_1, β_1) and (α_2, β_2) given by $(\alpha_1, \beta_1, \alpha_2, \beta_2) = (1.04, 89.45, 1.04, 36.82)$. These parameters correspond to mean and variance levels given by $(m_1, v_1^2, m_2, v_2^2) = (1.16 \times 10^{-2}, 1.30 \times 10^{-4}, 2.83 \times 10^{-2}, 7.69 \times 10^{-4})$. Weitzman's distribution corresponds to a gamma distribution with parameters $(1.78, 44.44)$. All represented distributions have the same mean and variance levels $(m, v^2) = (4 \times 10^{-2}, 9 \times 10^{-4})$.

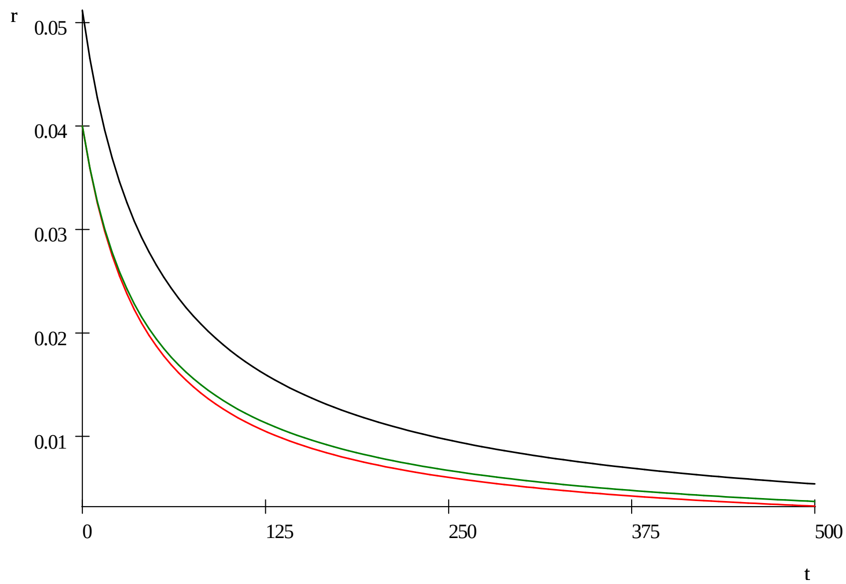

Figure 3: This figure represents the marginal discount rate curve $r_t = \sum_{i=1}^{N} \frac{w_i \rho_i \exp(-r^i t)}{\sum_{j=1}^{N} w_j \rho_j \exp(-r^j t)} r^i = \frac{\alpha_1+1}{\beta_1+t} + \frac{\alpha_2}{\beta_2+t}$ obtained through our calibration (upper curve) and compares it to the discount rate curve $r_t = \frac{\alpha}{\beta+t}$ of Weitzman (2001) (lower curve). The intermediate curve represents, with our calibration, the unweighted average $\sum_{i=1}^{N} \frac{w_i \exp(-r^i t)}{\sum_{j=1}^{N} w_j \exp(-r^j t)} r^i = \frac{\alpha_1}{\beta_1+t} + \frac{\alpha_2}{\beta_2+t}$. It is clear that the difference between our discount rate curve and Weitzman (2001)'s curve mainly results from the fact that, contrarily to the certainty equivalent approach, more impatient experts are more heavily weighted in the equilibrium approach.

Time period	Name	Numerical value	Approx. rate	Weitzman's num. value	Weitzman's appr. rate
Within years 1 to 5 hence	Immediate Future	4.99%	5%	3.89%	4%
Within years 6 to 25 hence	Near Future	4.23%	4%	3.22%	3%
Within years 26 to 75 hence	Medium Future	2.82%	3%	2.00%	2%
Within years 76 to 300 hence	Distant Future	1.50%	1.5%	0.97%	1%
Within years more than 300 hence	Far-Distant Future	0.16%	0%	0.08%	0%

Table 1 - Approximate recommended sliding-scale discount rates

This table compares for different time periods the recommended discount rates that result from our approach and those resulting from Weitzman (2001)'s approach. These rates are computed recursively. For the first period, we compute the rate that, if applied continuously from date 0 to the middle of the period would lead to the discount rate for that maturity. For next periods, we compute the rate that, if applied continuously from the beginning of the period to the middle of the period and compounded with the rates already computed for previous periods would lead to the discount rate for that maturity. The exact as well as approximate (recommended) results are then provided for both approaches.