Multiple Barrier Options Pricing

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Abstract

Nous étudions dans cet article de nouvelles solutions numériques, basées sur la technique de résolution de l'équation de la chaleur par séries de Fourier. Nous considérons, entre autres, plusieurs options, présentant des caractéristiques asiatiques, des clauses d'exercice anticipé et des clauses barrières, écrites sur un sous-jacent lognormal versant des dividendes discrètes et un taux continu de dividendes. Une fois énoncés les résultats généraux, nous nous intéressons au problème causé par l'existence de basses volatilités et démontrons son importance pratique. Enfin, nous exposons le mode de construction d'un arbre de Cox à pas de temps variable, nous montrons comment la technique sus-mentionnée peut être utilisé conjointement à la technique de Fourier pour évaluer des options bermudéennes complexes et obtenir a temps de calcul égal des prix plus justes.

Abstract

We first investigate new numerical solutions, based on the Fourier expansion heat equation solving technique, of numerous path dependent options written on lognormal assets paying discrete or continuous dividends. The foundations of the article having thus been laid, we give a rather general account of barrier options pricing with low volatilities, and explain why this issue has to be addressed. After having described how one can build multiple time-step Cox trees, we show that, using the Fourier technique together with the muti-time-step tree technique, one can drastically improve the level of accuracy when computing the price of simply path-dependent Bermudan barrier options.

Introduction

Barrier options are now widely used. Less costly than the equivalent plain vanilla options, they allow the singling out of a given risk and the corresponding hedging. However, computing the fair price of a barrier option is far more difficult than computing the fair price of its standard equivalent. Apart from the time consuming numerical simulations and the usual trees, the pricing schemes available today for valuing these options are: an extended Black & Scholes formula limited to single barrier options and a couple of recently deviced models offering some guidance for pricing double barrier options.

The two main drawbacks of these models are that they are limited to strictly European options written on stocks paying a continuous dividend yield and that they do not offer a framework which allows for more general barrier clauses than the double knock out one.

The aim of this paper is to describe a general framework which provides almost real time prices for simply path dependent Bermudan multiple barrier options, written on stocks paying discrete or continuous dividend yields.

The main mathematical tools used in this paper are the Fourier expansion heat equation solving technique and the classical change of variable through which the Black & Scholes PDE reduces to the heat equation.

The present paper is organized as follows:

In the first part of the paper we recall how the classical Fourier expansion technique can provide simple and CPU friendly solutions to the double barrier option pricing problem and we show how discrete dividends can be accounted for in this framework.

In the second part of the paper we show how our method can easily be applied to price simply path dependent options and truly complex options with combined multiple In and Out barriers.

In the third part of the paper we summarize the problems that may arrise when the underlying asset volatility reaches low levels and we offer some guidance on how to address those issues. Furthermore, we show how this can be done in a single coherent framework.

In the fourth part of the paper, we explain how a multi-time-step Cox tree can be built and we demonstrate how the Fourier expansion technique should

be combined with the so built-multi-time step Cox tree to obtain close to real time pricing and hedging solutions for those Bermudan claims that typically cannot be priced by any other method than the Cox tree discretization scheme — i-e. finite difference method.
Finally we carry out a numerical study on a fairly commonly traded asset: we give the price obtained with our technique, the price obtained with the usual Cox tree method and the limit tree price assumed to be true as well as the various computational times.

1 The Fourier expansion technique

1.1 Notations & Well know results

Throughout the paper:
• We place ourselves in a no arbitrage world and suppose given a risk free continous interest rate r and an asset volatility σ.
• We denote by S the price of a stock following a piecewise lognormal process with volatility σ; there are dates $T_0, T_1, \ldots, T_n$ such that $\forall i \in [0, n-1]$,

$$\frac{dS}{S} = (r-d)dt + \sigma dW_t$$

with d an eventually zero continous dividend yield.
• We denote by P the price of a contingent claim written on S with payoff $f(S)$ and maturity T.
It is a well known fact that P satisfies the Black & Scholes PDE :

$$\frac{\partial P}{\partial t} + (r-d)S\frac{\partial P}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 P}{\partial P^2} = rP$$

Setting

$$\alpha = \frac{1}{2} - \frac{r-d}{\sigma^2} \quad \text{and} \quad \beta = -(1-\alpha)^2 - \frac{2d}{\sigma^2}$$

and setting the variables

$$x = ln\left(\frac{S}{B_d}\right) \quad \text{and} \quad y = \frac{1}{2}\sigma^2(T-t)$$

where B_d is, for the time being, any non zero positive number, we obtain the existence of a function Φ such that

$$P(S,t) = e^{\alpha x + \beta y}\Phi(x,y) \quad \text{and} \quad \frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2}$$

1.2 The double barrier case

Let us now suppose that the contingent claim P pays a predetermined rebate and vanishes whenever S crosses an upper barrier B_u or a lower barrier B_d. Let R_u and R_d be these rebates. Set $\delta = ln\left(\frac{B_u}{B_d}\right)$.
Than $\forall x, y$ the function Φ satisfies the boundary conditions

$$\Phi(x,0) = f(B_d e^x) \quad \text{and} \quad \Phi(0,y) = R_d e^{-\beta y} \quad \text{and} \quad \Phi(\delta,y) = R_u e^{-\alpha\delta-\beta y}$$

We give the general solution of this equation in APPENDIX A. For the sake of simplicity, we shall here proceed with $R_u = R_d = 0$. This restriction is not motivated by theoretical considerations but adopted for the sake of perspicuity. Hence:

$$P(S,t) = e^{\alpha x+\beta y} \sum_{n=1}^{+\infty} \left(\frac{2}{\delta} \int_0^\delta f(B_d e^u) sin(\frac{n\pi}{\delta}u) e^{-\alpha u} du \right) sin(\frac{n\pi}{\delta}x) e^{-\left(\frac{n\pi}{\delta}\right)^2 y}$$

We give in APPENDIX B the value of the integral terms above when f is either the payoff of a call or a put.

1.3 Accounting for discrete dividends

When trying to value, in the proability theory based framework, a contingent claim written on a stock, one must assume continuously paid dividends. An assumption harmless when dealing with plain vanilla options but which can lead to large errors when one adds, for instance, a barrier clause.
Consider the case of a Down & Out option with maturity 1 year, written on a stock paying in one month a 5% dividend. Taking the spot and the strike both equal to 100, the barrier at 95, the risk free rate equal to 5% and the stock volatility equal to 20%, the true price of the option is about 2.5 whereas the continuous dividend yield price would be 3.8.

We shall, in this section, show how discrete dividends can be accounted for in the Fourier framework. Let T_0 be a date at which the share S strips off a dividend and denote by a the ratio

$$a = \frac{S_{T_0^+} - S_{T_0^-}}{S_{T_0^+}}$$

For example, if the share strips off a 5% dividend then

$$a = \frac{S_{T_0^+} - 0.05 S_{T_0^+}}{S_{T_0^+}} = 0.95$$

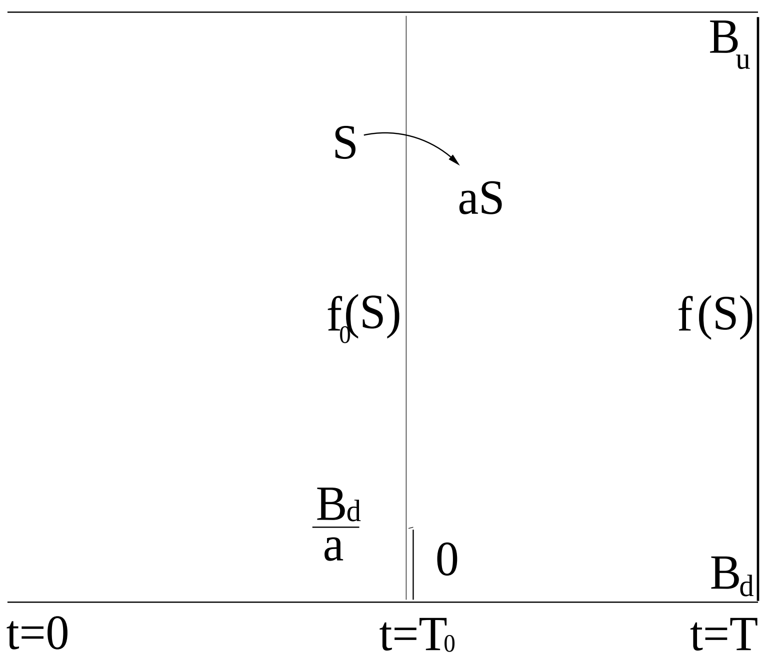
A key assumption in what follows is that the value of a and the date T_0 are both deterministic. Set

$$y_0 = \frac{\sigma^2}{2} T_0 \quad \text{and} \quad y_1 = \frac{\sigma^2}{2}(T_1 - T_0)$$

The trick is now to patch the option price at time T_0. Indeed, although the trajectories of the asset price are discontinuous, the trajectories of the contingent claim price are continuous as the dividend yield paid is supposed pretedermined — and therefore has been accounted for in the option price since the beginning of its life. Consequently at date T_0

$$P(S, T_0^-) = P(aS, T_0^+)$$

This very simple matching relation is the key to all that follows. Graphically:



Let $f_0(S)$ be the price at time T_0^- of the contingent claim P. Therefore

$$f_0(S) = P(S, T_0^-) = \begin{cases} P(aS, T_0^+) \text{ if } S > \frac{B_d}{a} \\ 0 \text{ if } S \leq \frac{B_d}{a} \end{cases}$$

Call Q the contingent claim that pays $f_0(S)$ at T_0 if none of the knock out barriers has been hit during the time interval $[0, T_0]$. Obviously, at any time $t < T_0$

$$Q(S,t) = P(S,t)$$

Therefore, to obtain the price of P at $t = 0$ we just need to apply, twice, the method described in the previous paragraph.
Denote by (c_n) the Fourier expansion coefficients of P on $[T_0, T]$ and by (d_n) the Fourier expansion coefficients of Q on $[0, T_0]$. These coefficients must satisfy the relation:

$$d_p = e^{\beta y_0 - \alpha\mu} \sum_{n=0}^{+\infty} c_n e^{-\left(\frac{n\pi}{\delta}\right)^2 y_0} \frac{2}{\delta} \int_\mu^\delta \sin\left(n\frac{\pi}{\delta}(u-\mu)\right) \sin\left(p\frac{\pi}{\delta}u\right) du$$

Where $\mu = -\ln(a)$. The value of the integral term is given in APPENDIX C. It should be noticed that this value decreases faster than $\mathcal{O}\left(\frac{1}{n^2}\right)$.

1.4 Numerical study

We consider here the case of an option with the following features:

- It is a double knock out call,
- Its period of maturity is twelve months,
- It is written on a stock paying, after six months, a predetermined discrete dividend expressed as a percentage of the share value.

Taking an annual constant 5% actuarial risk free rate, an annual constant 20% asset volatility, a spot value $S = 100$, we let the barriers and the percentage of dividends vary.

The computations having been performed on a SUN Ultra 2 computer with C double type precision, we obtain the following prices for an option at the money:

Div. Yield	Barriers	Monte Carlo	Fourier	Cox Tree	Nb. of points
0%	$\begin{cases} B_u = 120 \\ B_d = 80 \end{cases}$	1.114	1.114	1.124	36000
				1.127	24000
				1.136	12000
				1.140	6000
				1.164	1200
5%	$\begin{cases} B_u = 130 \\ B_d = 70 \end{cases}$	2.848	2.848	2.861	36000
				2.862	24000
				2.874	12000
				2.880	6000
				2.924	1200
10%	$\begin{cases} B_u = 140 \\ B_d = 60 \end{cases}$	3.429	3.429	3.436	36000
				3.438	24000
				3.443	12000
				3.448	6000
				3.466	1200

All the Fourier prices have been computed with the first 110 coefficients of the Fourier expansion. The "Nb. of points" parameter corresponds to the number of points on the Cox tree.The Fourier prices are obtained instantaneously with a totally satisfying level of accuracy.

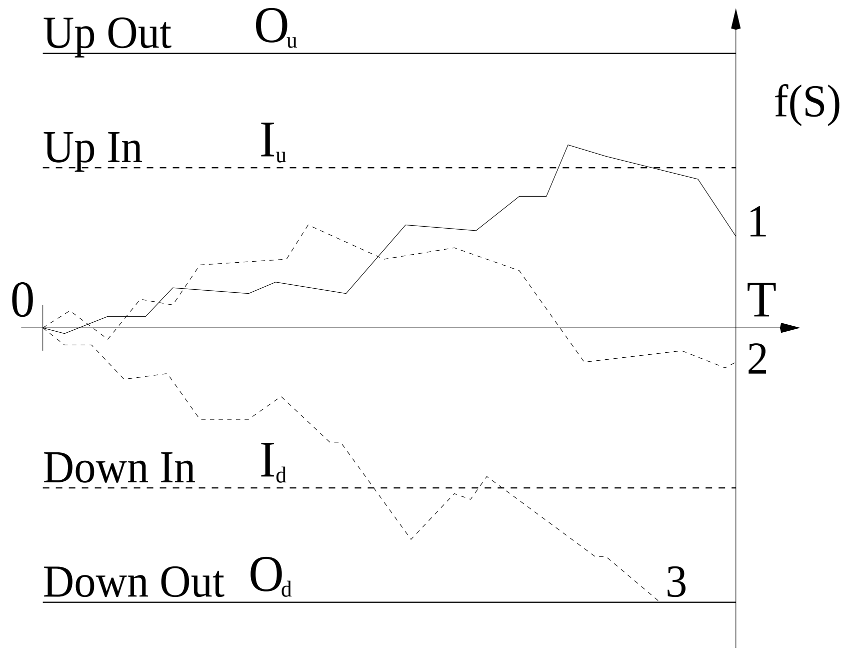
2 Worked out examples

We shall, in this section, give examples of how the same techniques can be used to price fairly complex options.
Options with In & Out barriers, although some of them are not commonly traded, can be very useful as they allow us to decompose more complex derivatives, where the kind of the payoff — e-g. the strike or the call/put feature — depends on whether a certain barrier has been crossed or not.

2.1 Combining In & Out Barriers

Let P be the price of an option with payoff $f(S)$ and 4 barriers: 2 In Barriers and 2 Out Barriers.
Precisely: the option pays $f(S)$ if at least one of the In Barriers has beeen

crossed and none of the Out Barriers has been crossed.
Graphically:



The trajectories 2 and 3 don't give rise to any payoff whereas the trajectory 1 pays $f(S)$ at date T.

We shall construct a function $\tilde{f}$ such that the price of our option is equal to the price of the double knock out option, with Down Barrier O_d and Up Barrier O_u, paying $\tilde{f}(S)$ at T.

In order to achieve this, let us introduce appropriate double knock out options. Let B_1 be the set of barriers $\{I_d, I_u\}$ and B_2 the set of barriers $\{O_d, O_u\}$.

We make the usual variable change:

$$x = \ln\left(\frac{S}{O_d}\right) \quad \text{and} \quad y_0 = \frac{\sigma^2}{2}T$$

Let us denote by $\rho_1(x_0, x)dx$ the probability that, starting at x_0, one ends up at T in the $[x, x+dx]$ interval without having crossed the barriers B_1 and by $\rho_2(x_0, x)dx$ the probability that, starting at x_0, one ends up at T in the $[x, x + dx]$ interval without having crossed the barriers B_2.

Setting $\delta_1 = \ln\left(\frac{I_u}{I_d}\right)$ and $\delta_2 = \ln\left(\frac{O_u}{O_d}\right)$, we see that for $i \in \{1,2\}$, $e^{-rT}\rho_i(x_0,x)dx$ is the price of the double knock out option with barriers B_i of which the pay-off at date T is 1 on the $[x, x+dx]$ interval and 0 elsewhere.

In APPENDIX D, we show, by using Dirac probabilities in the Fourier frame-work, that

$$\rho_i(x_0,x) = \frac{2e^{\alpha x+\beta y_0}}{\delta_i}\sum_0^{+\infty}\sin\left(\frac{n\pi}{\delta_i}x_0\right)\sin\left(\frac{n\pi}{\delta_i}x\right)e^{-\frac{n^2\pi^2}{\delta_i^2}y_0}$$

Now, a quite straigthforward analysis of the problem shows that:

- For $S \in [I_u, O_u]$ or $S \in [O_d, I_u]$, the option pays $f(S)$ at T.
- For $S \in [I_d, I_u]$, the option P pays $f(S)$ if one of the B_1 barriers has been crossed and none of the B_2 barriers has been crossed. The probability of this elementary event is $\rho_1(1-\rho_2)$.

Therefore, starting at $t=0$ with a spot price equal to S_0 the price P is the same as that of the double knock out option, with down barrier O_d and up barrier O_u, paying at date T

$$\tilde{f}(S) = \begin{cases} \rho_1\left(\ln\left(\frac{S}{0_d}\right)\right)\left(1-\rho_2\left(\ln\left(\frac{S}{O_d}\right)\right)\right)f(S) & \text{if } S \in [I_d, I_u] \\ f(S) & \text{if } S \in [O_d, I_d] \cup [I_u, O_u] \\ 0 & \text{otherwise} \end{cases}$$

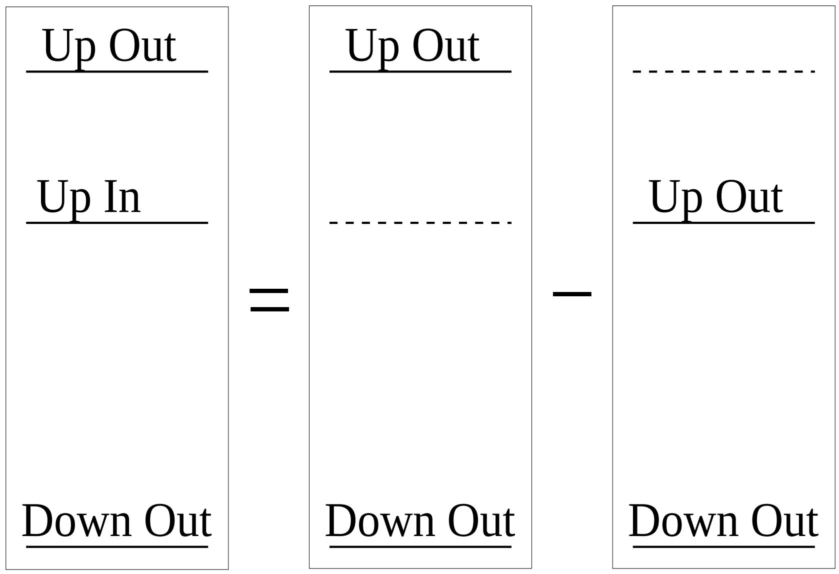
Therefore the price P at date 0 is

$$P(S_0) = \sum_0^{+\infty} d_n \sin\left(\frac{n\pi}{\delta_1}\right)\ln\left(\frac{S_0}{O_d}\right)$$

Where the coefficients d_n can either be computed numerically or with closed formulae, however it should be noted that the simple numerical procedure already ensures a high level of accuracy.

2.2 Simple "Double Knock Out" decomposition

Had there been no Down & In, or no Up & In, barrier clause on the previous option, its price could have been expressed quite simply as the difference of the price of two double knock out options. Graphically:



2.3 Simple path dependency

In the case of a forward start or of a chooser option, the final payoff is not solely a function of S but can be of the form $f(S_1, S)$ where S_1 is the value of S at date $T_1 \in [0, T]$.
Here, the patching principle can easily be extented by using two dimensionnal Fourier series.
Let P be the price of a double knock out European option paying $f(S_1, S)$ at date T with barriers B_d and B_u.
Let

$$x = \ln\left(\frac{S}{B_d}\right) \ , \ y_1 = \frac{\sigma^2}{2}(T - T_1) \ , \ y_0 = \frac{\sigma^2}{2}T_1 \ \text{and} \ x_1 = \ln\left(\frac{S_1}{B_d}\right)$$

In order to apply the patching principle at date T_1, let $f_1(S)$ be the price of the option at date T_1.

At date T_1, the payoff of the option is a deterministic function $P(S_1, .)$, therefore we can apply the Fourier expansion technique on the $[T_1^+, T]$ time interval. Denote by $c_n(x_1)$ the Fourier coefficients:

$$f_1(x) = e^{\alpha x + \beta y_1} \sum_{n=0}^{+\infty} c_n(x_1) \sin\left(\frac{n\pi}{\delta} x\right) e^{-\frac{n^2\pi^2}{\delta^2} y_1}$$

With

$$c_n(x_1) = \frac{2}{\delta} \int_0^{\delta} f\left(B_d e^{x_1}, B_d e^{x}\right) e^{-\alpha x} dx$$

Patching at date T_1^-, we obtain the Fourier expansion of P at date 0:

$$f(x) = e^{\alpha x + \beta y_0} \sum_{n=0}^{+\infty} c_n \sin\left(\frac{n\pi}{\delta} x\right) e^{-\frac{n^2\pi^2}{\delta^2} y_0}$$

With

$$c_p = \frac{4e^{\beta y_1}}{\delta^2} \sum_{n=0}^{+\infty} e^{-\frac{n^2\pi^2}{\delta^2} y_1} I(n,p)$$

Where

$$I(n,p) = \left(\int_0^{\delta} \int_0^{\delta} f\left(B_d e^{x_1}, B_d e^{x}\right) e^{-\alpha(x+x_1)} \sin\left(\frac{n\pi}{\delta} x\right) \sin\left(\frac{p\pi}{\delta} x_1\right) dx.dx_1 \right)$$

3 Low Volatility Related Issues

3.1 Generalities

It is a well known fact that whenever volatilities reach low levels, many normally satisfying pricing models start producing biased prices; thus, even the Black & Scholes formula for simple barrier options, when implemented in a naive way, does not converge for sufficiently low volatilities.

For the Fourier approach described in this paper (and even more so for the Kunitomo & al. approach) the problem is slightly more complex as the convergence of the scheme now depends both on the volatility and the time to maturity.

This limiting case problem may at first glance seem to be of little importance. There are however at least three fundamental reasons why it deserves to be addressed in a satisfactory way :

i) For very long term options on relatively low volality assets (e.g. a well diversified basket on the Toronto stock exchange).

ii) Although options may not be traded a few days before their expiry date, they still appear in the books and must therefore be hedged. Obviously, the most appropriate hedging strategy with respect to an option expiring in a short while is usually a rather intuitive one. However, a trader usually does not hedge options on an individual basis (as it would be both cumbersome and too expensive) but globally, as part of a portfolio. Therefore, if a significant mistake arises when computing the price of a product, however close this product may be to its maturity, it can have a dramatic effect on the hedge of the whole portfolio. A closely related phenomenon is that of the "near the barrier shooting up gamma", well known to traders.

iii) To get familiarized with a product, or to check the coherence of the price he is about to make, a trader does often perform a series of tests, choosing among a set of intuitive parameters (limiting cases from a mathematical point of view). Because any pricing theory is to some extent incomplete, as the perfect modelling of stock markets would also necessitate that of human behaviour, the process involving the trader's intuition is a key element of price making. Therefore, the tools used by a trader must not be biased.

3.2 Single Barrier Options Fixing Schemes

As we are here mainly concerned with the Fourier technique, we shall not linger too long on how the Black Scholes low volatility problem can be addressed.

Let it suffice to say that the commonly used "capped exponential":

$$\exp(x) \ := \ \text{IF } x < 100 \text{ THEN } \exp(x) \text{ ELSE } 0$$

is bad pratice. It is based, indeed, on the misleading and unjustified assumption that whenever an expression of the form $A(x) \to +\infty$ appears in a financial formula, the whole product $A(x)B(x)$ must tend toward zero to compensate for the divergence of A. A better and usually satisfying pratice (even with no more than the C double type precision) is to replace products of the form $A(x)B(x)$ by their mathematically (but not computer wise) perfectly equivalent expression $\exp(\ln(A(x)+\ln(B(x)))$. Surprisingly enough, even in large investment banks, some senior quantitative analysts seem not to know about this computer trick.

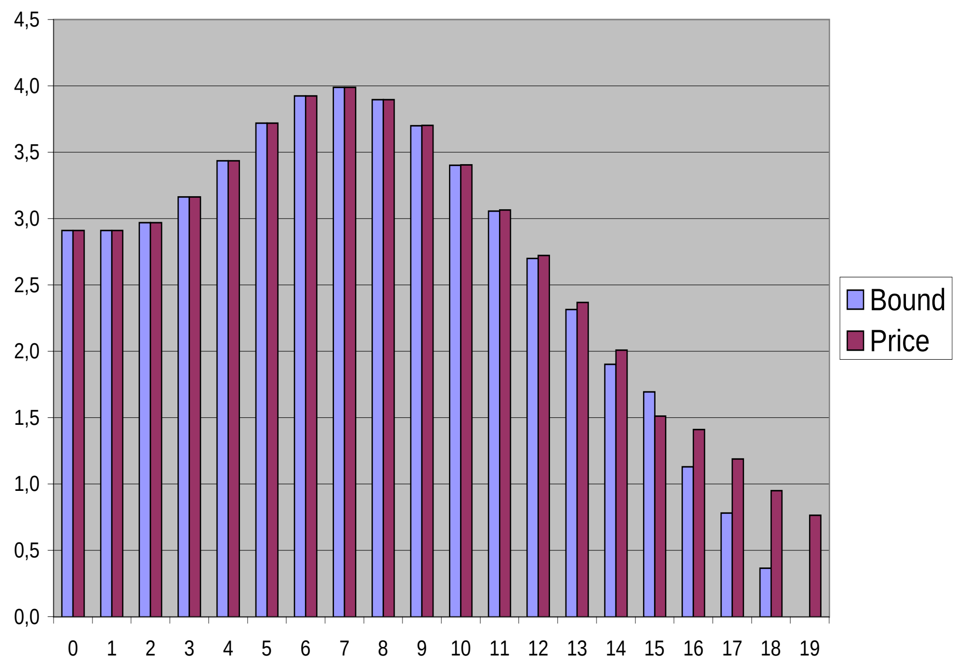
The single barrier problem being thus solved, let us again turn our attention to the double barrier problem which has so far been our main concern.

3.3 Double Knock-Out Options with Low Volatilities

The naive approach for finding an upper bound to a double barrier option price is to write:

$$\text{Bound} = \text{Single Up Out} + \text{Single Down Out} - \text{Vanilla}$$

Although this approach is sufficiently naive, it yields more than just a simple lower bound when the volatility is low and the barriers are "distant with respect to the volatility" (we shall define below what exactly we mean by distant). Indeed, in this special case, the events "*hitting the upper barrier*" and "*hitting the lower barrier*" are nearly disjoint so that what was previously only an lower bound now turns out to be a very accurate approximation !
The graph below shows both the price of a double knock out option and the value of the lower bound given above.



3.4 The Fourier Approach with Low Volatilities

With the usual C double type precision, only the first 15 digits of a number are significant. Therefore, with coefficients in the Fourier expansion as high as 10^{20}, it seems abnormal that we should be able to compute the function Φ. However, one must recall that the "real" coefficients involved in the computation of the function Φ are

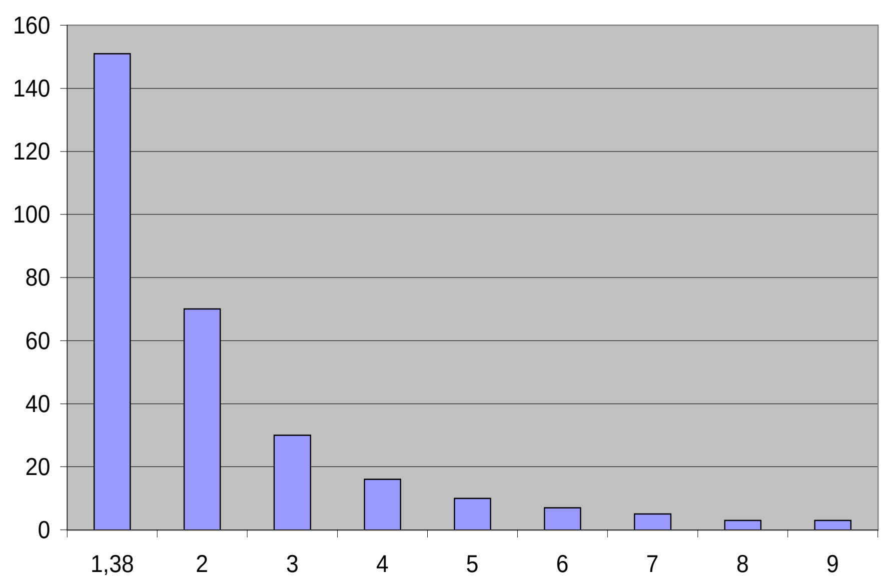
$$\widetilde{c_n} = c_n e^{-\left(\frac{n\pi}{\delta}\right)^2 T}$$

So that for all integers n,

$$|\widetilde{c_n}| \leq |c_n| e^{-\left(\frac{\pi}{\delta}\right)^2 T}$$

Therefore, the smaller theδ (i.e: the closer the barriers) the better the convergence.

The graph below shows how $(Sup(\widetilde{c_n}))$ vary with volatility for a double knock out option.



The price of an option with an unattainable barrier must naturally be very close to the plain vanilla Black & Scholes price.
However, if the barrier is very high, using the simple barrier Black & Scholes formula may yield, as a result of computer precision limitations, a price less accurate than the one that would have been obtained with the plain vanilla Black & Scholes formula by simply cancelling out the barrier.

The above discussion, although an over simplification of what usually occurs in financial markets, has enabled us to emphasize an essential point: in order to improve the accuracy of the prices computed (and therefore of the hedge parameters for which traders often require a "by perturbation" calculation) it may be necessary to artificially change the position of the barrier for the purpose of pricing.

To know whether a given barrier displacement is allowable or not, many subbtle approaches may be used. We shall now briefly sketch an approach which, although mathematically very simple, gives a fair idea of how a barrier-spread reduction algorithm can be written.

Assuming positive interest rates and denoting by F the forward at maturity, the probability that the asset price be above the barrier U at maturity is given by:

$$P_U^+ = \frac{1}{\sqrt{2\pi}\sigma} \int_{\frac{U}{F}}^{+\infty} e^{-\frac{x^2}{2\sigma^2 T}} dx = \mathcal{O}\left(e^{-\frac{U^2}{2\sigma^2 F^2 T}}\right)$$

Therefore, for an option with bounded payoff, the maximum error that can be introduced by displacing the upper barrier to U is

$$\mathcal{O}\left(e^{-\frac{U^2}{2\sigma^2 F^2 T}}\right)$$

Thus, K being the constant associated with the above $\mathcal{O}$ notation, given a desired precision ϵ on the price of the option, we can determine the lowest satisfactory upper barrier U by solving:

$$\epsilon = e^{-\frac{U^2}{2\sigma^2 F^2 T}}$$

i.e.

$$U = \frac{F\sigma\sqrt{T}}{2} \ln\left(\frac{\epsilon}{K}\right)$$

The same can be done for the lower barrier by replacing, in the above formula, the forward F by the spot S.

3.5 A convergence criterion based on the value of α

Let us consider a Double Knock Out call Option. Looking at the expression of the coefficients c_n given in APPENDIX B, we see that:

$$|c_n| = \mathcal{O}\left(\frac{e^{|\alpha|\delta}}{|\alpha|}\right)$$

Thus, C being the maximum permissible value of any of the $\widetilde{c}_n$ coefficients, a sufficient condition for the Fourier expansion to converge is given by:

$$Ce^{\frac{\pi}{\delta}T} \geq \frac{e^{|\alpha|\delta}}{|\alpha|}$$

Furthermore, $|\alpha|$ being, for σ low enough, a strictly decreasing function of σ, we have given a sense to the expression "the barriers are distant with respect to the volatility".

3.6 Concluding remarks on the low volatility case

We have shown above that although low volatilities pose a serious problem when one restricts oneself to the usual formulae, the difficulty can be overcome and a general purpose pricing methodology can be obtained by applying simple rules.

Let us offer a simple recipe:

1) If the $\widetilde{c}_n$ coefficients are all whithin a permissible range, compute the price of the option, otherwise:

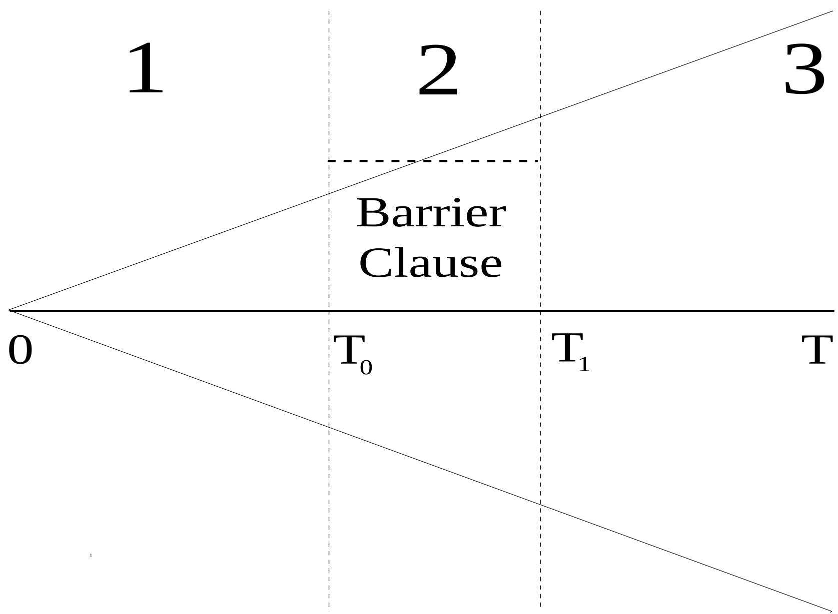
2) Try to tighten the barrier-spread, then compute the new $\widetilde{c}_n$ coefficients. If they are all whithin the permissible range, compute the price of the option, otherwise:

3) Use the disjoint events approximation.

4 Multi-time-step Cox Tree

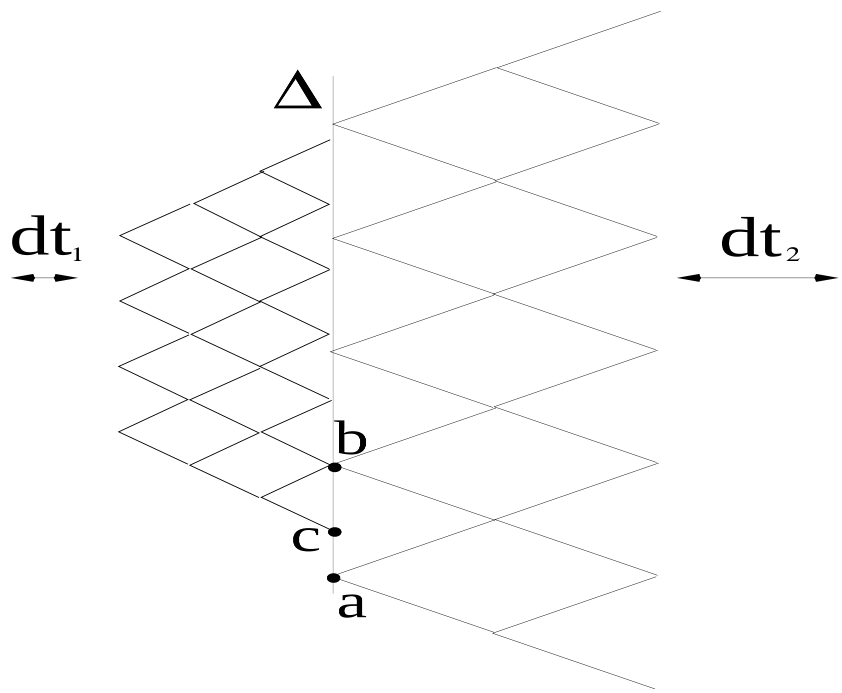
4.1 Framework

The patching principle described above can also be applied in discrete cases. This principle can for instance lead to the construction of multi-time-step binomial trees. Indeed, when discretizing a given situation, one does not need the same amount of precision at all computational times.
Let us for instance consider the case of an american option with a knock out barrier on a time interval of which the length is small with respect to the maturity of the option.



It is well know that trees are short sighted with respect to barrier clauses. Therefore one needs a much higher precision on the time interval 2 during which the barrier clause is alive than on the simple american time intervals 1 and 3.
On the classical Cox-Rubinstein tree, the time step is set once and for all by the recombining equation.

However, one can choose as many time steps as one desires if one decides, instead, to slice the Cox tree and patch the different slices following the above mentioned principle.



The usual recombining condition imposes that any node of the first tree lying on the interface Δ should coincide with a node of the second tree. In the usual case where σ and r are everywhere constant, this leads immediately to $dt_1 = dt_2$.
Therefore, to allow for $dt_1 \neq dt_2$ we shall instead interpolate logarithmically the price on the interface. Hence

$$P(c) = \exp\left(\frac{\log(S_b) - \log(S_c)}{\log(S_b) - \log(S_a)}\log(P(a)) + \frac{\log(S_c) - \log(S_a)}{\log(S_b) - \log(S_a)}\log(P(b))\right)$$

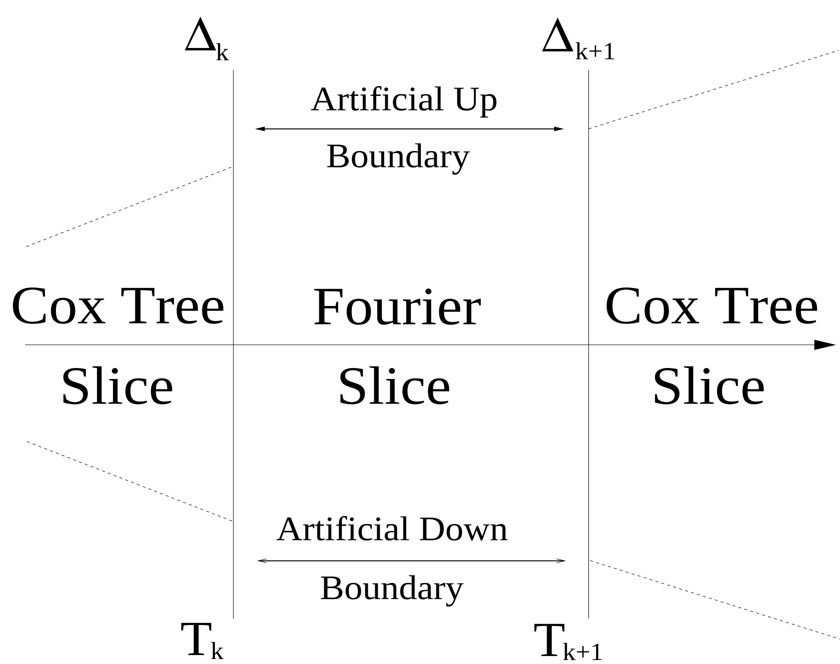
Where S denotes the price of the asset and P the price of the contingent claim being priced. The negative value case is dealt with similarly.

We have obviously made an approximation. However, one should keep in mind that only at the limit does the binomial distribution become gaussian

— therefore even when one chooses to patch perfectly the nodes on the interface, one only gets an approximate solution — and that the bias by which we replace the systematic and *cumulative* error, due to not seeing a barrier, is similar to that of the integration method by rectangular quadrature.

4.2 Mixed discrete and continuous methods

The only way to price an American option is usually to discretize the situation with some type of a finite difference scheme. Therefore, one must also use such a method to price a Bermudan option. However, pricing the whole option with a finite difference scheme would involve a loss both of time and accuracy.
Therefore, we prefer to combine the discrete and the continuous methods by applying our patching principle.
Suppose that P is the price of a Bermudan option written on an asset S and choose N dates $T_0, T_1, \ldots, T_N$ such that on any time interval $[T_k, T_{k+1}]$ the option P is either European or American. And suppose that k is an integer such that P is American on the intervals $[T_{k-1}, T_k]$ and $[T_{k+1}, T_{k+2}]$ and European on the interval $[T_k, T_{k+1}]$.



Brief sketch of the algorithm :

• On Δ_{k+1} — the interface at date T_{k+1} — interpolate logarithmically P

• Find, if necessary, two artificial boundaries, such that the probability of the asset price staying between these boundaries is suffitiently close to zero

• Compute the significant Fourier coefficients on Δ_{k+1}

• With the Fourier expansion, compute the price P on the interface Δ_k

• Given these values, run backward through the Cox tree slice ending at date T_k

During this process, the loss of precision, if any, won't occur during the Fourier $[T_k, T_{k+1}]$ slice but will be entirely due to the tree structure and thus, could not have been dealt with any better in the classical finite difference scheme — e-g. Cox tree.

Indeed, the precision of the result obtained on the Δ_k interface depends only on having selected the relevant set of harmonics on the Δ_{k+1} interface — e-g. with the Parseval relation

$$\frac{B_d^2}{\delta}\int_0^\delta f(B_d e^x)^2 e^{-2\alpha x}dx = \sum_{n=0}^{+\infty} c_n^2$$

one can determine with a small extra computationnal cost a value of N such that $\sum_{n=N}^{+\infty} c_n^2$ is as small as desired. Hence, the loss of precision on the Fourier slice is perfectly controlled.

Furthermore, at the same confidence level, the Fourier slice is always one order faster than the equivalent Cox slice. Hence by using this method one saves time in two different ways — by saving on the Cox trees on the European time intervals and by choosing the most efficient time steps on the American time interval — and one controls much more closely the accuracy of the result.

It should also be noticed that on the left side interface of a Fourier slice, the greek parameters — delta, gamma and theta — can be directly obtained from the Fourier expansion.

4.3 Numerical study of a Bermudan Option

We consider here the case of a commonly traded Bermudan option with the following characteristics:

• It is a double knock out call,

• Its period of maturity is twelve months. The option is European during the first 11 months of its life. During the last month, the option becomes American.

One can easily understand what makes such an option attractive to investors:

• The American feature on the latter portion of its life makes this option less vulnerable to sudden changes in the asset price,

• The price lowering effect of the barriers makes the option less costly; the choice of the barriers being directly related to the risk exposure the option buyer is willing to face.

Making use of the above described method, we proceed as follows:

• On the last month, we compute the price of the option on a Cox tree slice, with the usual backward induction process.

• On the first 11 months of the option life, we apply the Fourier technique with logarithmic interpolation.

Taking an annual constant 5% actuarial risk free rate, a spot value $S = 100$ and letting the volatility and the barriers vary, we obtain the following prices for an option at the money:

With $\sigma = 10\%$,

Barriers	Limit Price	Fourier Price	Pure Cox Price	Nb. of points
$\begin{cases} B_u = 110 \\ B_d = 90 \end{cases}$	0.923	0.923	0.937	12000
		0.922	0.941	6000
		0.922	0.965	1200
$\begin{cases} B_u = 120 \\ B_d = 80 \end{cases}$	4.24	4.24	4.26	12000
		4.24	4.27	6000
		4.24	4.28	1200
$\begin{cases} B_u = 130 \\ B_d = 70 \end{cases}$	6.185	6.19	6.19	12000
		6.18	6.19	6000
		6.17	6.20	1200

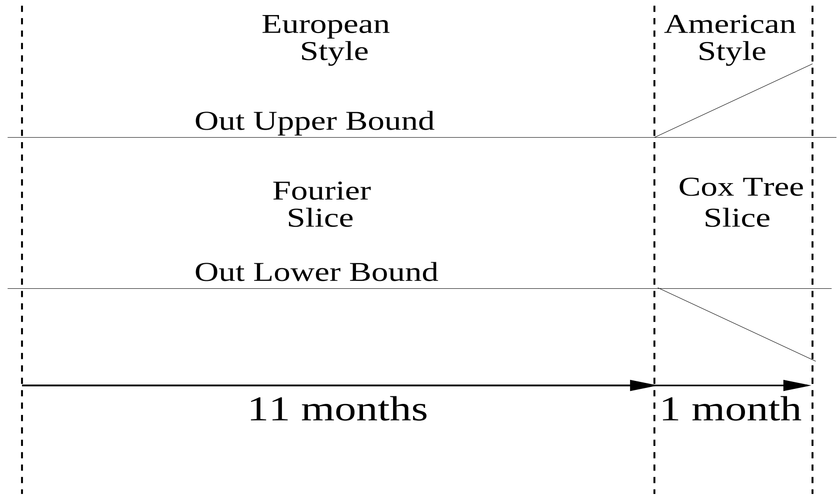
With $\sigma = 20\%$,

Barriers	Limit Price	Fourier Price	Pure Cox Price	Nb. of points
$\begin{cases} B_u = 120 \\ B_d = 80 \end{cases}$	1.55	1.55	1.57	12000
		1.55	1.58	6000
		1.55	1.63	1200
$\begin{cases} B_u = 130 \\ B_d = 70 \end{cases}$	3.97	3.97	4.00	12000
		3.97	4.01	6000
		3.95	4.04	1200
$\begin{cases} B_u = 140 \\ B_d = 60 \end{cases}$	6.35	6.35	6.37	12000
		6.34	6.39	6000
		6.34	6.40	1200

With $\sigma = 30\%$,

Barriers	Limit Price	Fourier Price	Pure Cox Price	Nb. of points
$\begin{cases} B_u = 130 \\ B_d = 70 \end{cases}$	2.03	2.03	2.06	12000
		2.03	2.07	6000
		2.03	2.09	1200
$\begin{cases} B_u = 140 \\ B_d = 60 \end{cases}$	3.97	3.97	4.00	12000
		3.96	4.03	6000
		3.95	4.08	1200
$\begin{cases} B_u = 150 \\ B_d = 50 \end{cases}$	6.02	6.01	6.05	12000
		6.01	6.05	6000
		6.00	6.15	1200

All the Fourier prices have been computed with the first 90 coefficients of the Fourier expansion. The "Nb. of points" parameter corresponds to the number of points on the Cox tree, with respect to the Cox method, and to the number of points on the last vertical row of the Cox tree slice, with respect to the Fourier technique.
Hence, with the same number of points n, the Fourier technique works about 10 times faster than the usual Cox tree method for large values of n and about 5 times faster than the usual Cox tree method for small values of n.
Eventually, the Fourier technique provides in less than 1 second a price with a level of accuracy corresponding to a 35 seconds computation using the usual Cox tree method.


European
Style
American
Style
Out Upper Bound
Fourier
Slice
Cox Tree
Slice
Out Lower Bound
11 months
1 month

5 APPENDICES

5.1 APPENDIX A: the rebates

The price of the double knock out option with rebates must be equal to the sum of the price of the double knock out option without rebates and the price of pure rebates — i.e. rebates payment if the barriers are touched and a zero payoff at maturity.
Let us denote by $e^{\alpha x+\beta y}\Phi(x,y)$ the price of the pure rebates. The function

$$\Psi_0(x,y)=\Phi(x,y)-e^{-\beta y}R_d\frac{\sinh(\sqrt{-\beta}(\delta-x))}{\sinh(\sqrt{-\beta}\delta)}-e^{-\beta y}e^{-\alpha\delta}R_u\frac{\sinh(\sqrt{-\beta}x)}{\sinh(\sqrt{-\beta}\delta)}$$

Is a solution of the heat equation and satisfies the boundary conditions

$$\Psi_0(0,y)=0 \quad \text{and} \quad \Psi_0(\delta,y)=0$$

Therefore, denoting by (d_n) the Fourier coefficients of the function

$$x\mapsto R_d\frac{\sinh(\sqrt{-\beta}(\delta-x))}{\sinh(\sqrt{-\beta}\delta)}+e^{-\alpha\delta}R_u\frac{\sinh(\sqrt{-\beta}x)}{\sinh(\sqrt{-\beta}\delta)}$$

One has :

$$\Psi_0(x,y)=\sum_{n=0}^{\infty}c_n\sin\left(\frac{n\pi}{\delta}x\right)e^{-\left(\frac{n\pi}{\delta}\right)^2y}$$

Therefore

$$\Psi(x,y)=e^{-\beta y}R_d\frac{\sinh(\sqrt{-\beta}(\delta-x))}{\sinh(\sqrt{-\beta}\delta)}+e^{-\beta y}e^{-\alpha\delta}R_u\frac{\sinh(\sqrt{-\beta}x)}{\sinh(\sqrt{-\beta}\delta)}$$
$$-\sum_{n=0}^{\infty}c_n\sin\left(\frac{n\pi}{\delta}x\right)e^{-\left(\frac{n\pi}{\delta}\right)^2y}$$

5.2 APPENDIX B: Call or Put

Let $\epsilon=\ln\left(\frac{K}{B_d}\right)$,

- The Fourier expansion coefficients of the call price are given by:

$$c_n=\frac{2}{\delta}\int_{\epsilon}^{\delta}(B_de^x-K)e^{-\alpha x}\sin\left(\frac{n\pi}{\delta}x\right)dx$$

$$= (-1)^n e^{-\alpha\delta}\left(\frac{2\delta(1-\alpha)e^{\delta}B_d}{(1-\alpha)^2\delta^2+n^2\pi^2} - \frac{2\delta\alpha K}{\delta^2\alpha^2+n^2\pi^2}\right)$$

$$+e^{-\alpha\epsilon}\left(\frac{n\pi}{\delta}\cos\left(\frac{n\pi}{\delta}\right) - \sin\left(\frac{n\pi}{\delta}\right)\right)\left(\frac{2\delta(1-\alpha)e^{\epsilon}B_d}{(1-\alpha)^2\delta^2+n^2\pi^2} - \frac{2\delta\alpha K}{\delta^2\alpha^2+n^2\pi^2}\right)$$

- The Fourier expansion coefficients of the put price are given by:

$$c_n = \frac{2}{\delta}\int_0^{\epsilon}(K - B_d e^x)e^{-\alpha x}\sin\left(\frac{n\pi}{\delta}x\right)dx$$

$$= \frac{2n\pi}{\delta}\left(\frac{2\delta(1-\alpha)B_d}{(1-\alpha)^2\delta^2+n^2\pi^2} - \frac{2\delta\alpha K}{\delta^2\alpha^2+n^2\pi^2}\right)$$

$$-e^{-\alpha\epsilon}\left(\frac{n\pi}{\delta}\cos\left(\frac{n\pi}{\delta}\right) - \sin\left(\frac{n\pi}{\delta}\right)\right)\left(\frac{2\delta(1-\alpha)e^{\epsilon}B_d}{(1-\alpha)^2\delta^2+n^2\pi^2} - \frac{2\delta\alpha K}{\delta^2\alpha^2+n^2\pi^2}\right)$$

5.3 APPENDIX C: Dicrete dividends

One has:

$$\int_{\mu}^{\delta}\sin\left(n\frac{\pi}{\delta}(u-\mu)\right)\sin\left(p\frac{\pi}{\delta}u\right)du$$

$$= \frac{1}{2}\int_{\mu}^{\delta}\left(\cos\left((n-p)\frac{\pi}{\delta}u - \frac{n\pi}{\delta}\mu\right) - \cos\left((n+p)\frac{\pi}{\delta}u - \frac{n\pi}{\delta}\mu\right)\right)du$$

$$= \begin{cases} \frac{2\delta}{\pi(n^2-p^2)}\left(p(-1)^{n+p+1}sin\left(\frac{n\pi\mu}{\delta}\right) + n\sin\left(\frac{p\pi\mu}{\delta}\right)\right) & \text{if } p \neq n \\ (\delta-\mu)\cos\left(\frac{n\pi\mu}{\delta}\right) + \frac{\delta}{n\pi}sin\left(\frac{n\pi\mu}{\delta}\right) & \text{if } p = n \end{cases}$$

5.4 APPENDIX D: Computing probability distributions

Let f bet the function worth 1 on the $[x, x+dx]$ interval and 0 elsewhere. Consider the double knock out option with up barrier I_u and down barrier I_d paying f at date T. Denoting by x_0 the spot, the option price can be expressed as the sum of the following Fourier expansion:

$$\text{Price} = e^{\alpha x_0 + \beta y}\sum_{n=0}^{+\infty} c_n \sin\left(\frac{n\pi}{\delta_1}\right) e^{\frac{-2y^2\pi^2}{\delta_1^2}}$$

where c_n stands for the n-th Fourier coefficient computed as usual with

$$c_n = \frac{2}{\delta_1} \int_0^{\delta_1} f(I_d e^u) e^{-\alpha u} \sin\left(\frac{n\pi u}{\delta_1}\right) du$$

Whence

$$c_n = \frac{2}{\delta_1} e^{-\alpha x} \sin\left(\frac{n\pi x}{\delta_1}\right) dx$$

Besides

$$e^{-rT} \rho_i(x_0, x) dx = \text{Price}$$

Therefore

$$\rho_1(x_0, x) = \frac{2e^{\alpha x + \beta y_0}}{\delta_1} \sum_0^{+\infty} \sin\left(\frac{n\pi}{\delta_1} x_0\right) \sin\left(\frac{n\pi}{\delta_1} x\right) e^{-\frac{n^2\pi^2}{\delta_i^2} y_0}$$

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