Reconsidering asset allocation involving illiquid assets

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Abstract:

Alternative assets are gaining increasing importance in investor's portfolios. One of their defining characteristic is their poor liquidity which often translates into an inherent smoothing process of the returns. For asset allocation purposes, this feature has to be seriously addressed as it leads to a severe underestimation of the variance of returns and their correlation with other (standard) assets. In this article, in order to deal with practical issues, we extend previous researches which model the smoothing process as a moving average one in several directions: (i) we propose a correction for the case of numerous illiquid assets ; (ii) we investigate the implications of the standard practice of fitting autoregressive models in place of moving average models for the correction of the returns variance; (iii) we provide generalization to the case where the returns process is jointly governed by smoothing and true (economically) time-dependent behaviour. All the theoretical results are illustrated empirically with applications to US real estate and venture capital indexes.

Introduction

Alternative assets, that is assets which are different from core assets such as money market, bonds and equities, are gaining increasing importance in investor's portfolios; see for instance the recent survey of institutional practices by Strong, O'Sullivan and Cunningham [2003] or the almost doubling of assets under management of the hedge fund industry between 2000 and 2005, reaching the Bns $ 1,000 threshold.

One of the major features of alternative investments is that they are less liquid than standard ones, making them harder to valuate. Since by definition illiquid assets cannot be exchanged on a secondary market, one often relies on expert valuations to price them. The matter is then that prices may Table properties which are largely spurious. In particular, since returns are implicitly smoothed, they appear far less volatile and less correlated with other assets than they are in reality. To simply illustrate this problem, let us imagine that an investor is asked to choose among two funds. The strategy of both funds is simple as it consists in replicating the S&P index performance and we assume that both funds respond to this assignment perfectly. The difference is that the first fund (say A) is valuating its portfolio every day on the basis of the true contemporaneous S&P return while the second fund (say B) reports the rolling average return over the past month. Let us imagine now that the investor computes the standard deviation of the returns of both funds and their correlation with the S&P index over the period 1995-2004 with daily data. For Fund A (the one which uses contemporaneous information), she will find that these statistics are 18% p.a. and 100%, respectively. For Fund B (the one which uses moving average of returns), she will infer tiny figures such as a 3% p.a. standard deviation and a 20% correlation with the S&P index returns. With such statistics, we have no doubt that the investor will prefer Fund B. What is troublesome is that she will probably do that even in the case where fees are larger for Fund B while this Fund adds no value. Contrary to what she infers from data, the investor will have realized no risk reduction or diversification gains with Fund B since it will remain exposed to the same risk factors as the ones of Fund A or its existing equity portfolio.

Despite being only a caricature, the previous example perfectly illustrates the dangers of the naïve use of standard techniques when it comes to evaluate the statistical properties of an illiquid asset, in the sense of its returns being smoothed, as it typically leads to incorrect inference of allocation among assets with too large a weight on the illiquid asset. As the proportion of alternative assets in portfolios is mounting, this implication is increasingly problematic and must be treated with care. One can find in the literature several attempts to tackle this matter, starting notably with the work of

Geltner [1991, 1993] on real estate indexes. In a recent paper, Getmanski, Lo and Makarov [2005] (GLM henceforth) have formalized this issue in an elegant and precise way. In their framework, the smoothing process is recovered either through the autocorrelation structure of the observed returns or through a regression of observed returns on contemporaneous and lagged values of a factor. The authors then propose an application to (single) hedge funds returns, which are probably the less illiquid of illiquid assets.

In this paper, we extend this research to deal with issues which are of crucial importance, both for researchers and practionners. More precisely, we analyze three key points. First, what happens when one faces two (or more) illiquid assets, such as a portfolio mixing Private Equity and real estate? Second, what are the implications of the usual practice which consists in fitting autoregressive (AR) processes on observed returns while smoothing implies a moving average (MA) behaviour? As we shall see below, the correction of standard deviation and the correlation with other assets can be rather sensitive to this hypothesis. Third, how to consider the case where the true (economic) process governing the asset prices is itself autocorrelated? We argue that this point is particularly plausible in the case of real estate and, again, the correction of statistics turns out to be very sensitive to this hypothesis. In each case, we propose solutions and methods which are easy to implement in practice and illustrate them with the case of US real estate and venture capital. After introducing some notations and briefly reviewing the GLM approach, the paper successively deals with each of these three issues. We end our paper by some concluding comments and an example of its various results with a typical asset allocation problem.

berNotations and theoretical background

We begin by giving a brief recall of the GLM framework[1]. Let R_t denote the (continuously compounded) true return of an alternative asset with mean μ and variance σ^2. It is assumed that this true (or effective) return is not observed but rather that the observed return at time t is a weighted average of past and present returns:

$$R_t^o = \sum_{i=0}^{k} \theta_i R_{t-i} \ , \tag{1}$$

thus implying a smoothing process. The authors further impose the restrictions $0 \leq \theta_i \leq 1 \ \forall i$ and $\sum_{i=0}^{k} \theta_i = 1$ so that all the information about R_t can be inferred from the time series of R_t^o. From this set of hypothesis, it is straightforward to deduce first that the average return is unchanged,

$E\left[R_t^o\right] = E\left[R_t\right] = \mu$, meaning that the true value of the asset is revealed on average which is logical since smoothing based on a moving average does not imply any bias in the first moment. On the contrary, smoothing leads to an understatement of the variance, since:

$$\mathrm{var}\left[R_t^o\right] / \mathrm{var}\left[R_t\right] = \sum_{i=0}^{k} \theta_i^2 \leq 1 . \tag{2}$$

The limit case of no-bias is obtained for $\theta_0 = 1$ and $\theta_i = 0 \ \forall i > 0$, that is with no smoothing. As a result, the Sharpe ratio is overestimated, one implication already clearly put forward in Lo [2002]. The quantity $\sum_{i=0}^{k} \theta_i^2$, known as the Herfindahl index in industrial organisation economics, can act as a measure of the unsmoothness of the process. It fluctuates between $1/(k+1)$ when all θ_i are equal and 1 when there is no smoothing. Smoothing also has implications for the analysis of correlation. Adding the hypothesis that the true returns are not serially correlated, both for the illiquid assets and another asset which is assumed to be liquid and whose returns are denoted by F_t, GLM shows that :

$$\mathrm{corr}\left(R_t^o ; R_{t-j}^o\right) / \mathrm{corr}\left(R_t ; R_{t-j}\right) = \sum_{i=0}^{k-j} \theta_i \theta_{i+j} \Big/ \sum_{i=0}^{k} \theta_i^2 \quad \text{for } 0 \leq j \leq k , \tag{3}$$

$$\mathrm{corr}\left(R_t^o ; F_{t-j}\right) / \mathrm{corr}\left(R_t ; F_{t-j}\right) = \theta_j \Big/ \sqrt{\sum_{i=0}^{k} \theta_i^2} \quad \text{for } 0 \leq j \leq k , \tag{4}$$

and 0 elsewhere. Equation (3) shows that some spurious autocorrelation is induced for $j > 1$. Equation (4) states that spurious lead-lag correlation is also introduced while the contemporaneous correlation is understated. This last result, coupled with the one in (2), has obviously severe consequences for asset allocation issues.

To illustrate the various theoretical results of this paper, we choose two important alternative assets: (i) venture capital quarterly returns observed from 1986Q3 to 2003Q4, as reported by Venture Economics; (ii) real estate quarterly returns observed from 1978Q1 to 2004Q3, as reported by the NCREIF indexes for various regions of the US (National, West, South, East, Midwest)[2]. We also need to choose a factor (or liquid asset) for each. Preliminary investigations showed us that the best results were obtained with the Nasdaq Composite index for venture capital and the 30-years Fannie Mae mortgage rates for real estate. In Table 1, we report descriptive statistics for the returns of alternative assets. All the real estate indexes are characterized by a large number of significant autocorrelations, with none being negative up to the 12th lag. While we will show later that some other reasons might also be invoked, there is clear presumption of a smoothing process for these indexes since they are typically drawn from the appraisal returns reported by the real estate experts. The pattern is somehow different for Venture Capital returns, for which the correlation with the factor is far higher, but also the smoothing behaviour is far less evident since if the first three autocorrelation coefficients are significant at the 1% level, there is no other positive significant autocorrelation and all coefficients are negative from lag 5 up to 12.

The smoothing structure is analyzed in more details in Table 2 where we present the results of the estimation of the smoothing coefficients θ_i through MA models. First, one estimates the MA(k) process, $R_t^o = \alpha + \varepsilon_t + \vartheta_1 \varepsilon_{t-1} + \ldots + \vartheta_k \varepsilon_{t-k}$, and then simply deduces the smoothing parameters as $\hat{\theta}_0 = 1/\left(1 + \hat{\vartheta}_1 + \cdots + \hat{\vartheta}_k\right)$, $\hat{\theta}_i = \hat{\vartheta}_i / \left(1 + \hat{\vartheta}_1 + \cdots + \hat{\vartheta}_k\right)$ $i > 1$ [3]. Without any surprise given the correlation structure, each time-series is associated with a significant MA process. The maximum lag associated to the MA model is higher in the case of real estate returns, notably for the national index, which is consistent with a smoother profile, as also confirmed by lower values of the Herfindahl index[4]. In the bottom of the table, we present the corrections for both the standard deviation of returns and the contemporaneous correlation with the factor[5]. Both statistics are obviously corrected upwards when the smoothing pattern is removed from the time-series.

The case of several illiquid assets

We begin our extensions of the framework by considering the case of multiple illiquid assets. Let us assume that there are two illiquid assets and that one wants to estimate their correlation. Both assets are subject to a smoothing process with coefficients $\theta_{i,(1)}$ and $\theta_{i,(2)}$ respectively. k_1 and k_2 denote the respective lags of both processes. Using the same set of assumptions as before, we infer that:

$$\mathrm{corr}\left[R_{t,(1)}^o ; R_{t,(2)}^o\right] = \frac{\sum_{i=0}^{\min(k_1,k_2)} \theta_{i,(1)} \theta_{i,(2)}}{\sqrt{\sum_{i=0}^{k_1} \theta_{i,(1)}^2} \sqrt{\sum_{i=0}^{k_2} \theta_{i,(2)}^2}} \mathrm{corr}\left[R_{t,(1)} ; R_{t,(2)}\right]. \quad (5)$$

From Cauchy-Schwartz inequality and the fact that $\theta_{i,(1)} \geq 0,\ \theta_{i,(2)} \geq 0\ \ \forall i$, we infer that:

$$\sum_{i=0}^{\min(k_1,k_2)} \theta_{i,(1)} \theta_{i,(2)} \leq \sqrt{\sum_{i=0}^{\min(k_1,k_2)} \theta_{i,(1)}^2} \sqrt{\sum_{i=0}^{\min(k_1,k_2)} \theta_{i,(2)}^2} \leq \sqrt{\sum_{i=0}^{k_1} \theta_{i,(1)}^2} \sqrt{\sum_{i=0}^{k_2} \theta_{i,(2)}^2} . \quad (6)$$

From (5) and (6), we deduce that in most cases, the correlation between both assets will be understated. The exception to this is obtained when $\sum_{i=0}^{\min(k_1,k_2)} \theta_{i,(1)} \theta_{i,(2)} = \sqrt{\sum_{i=0}^{k_1} \theta_{i,(1)}^2} \sqrt{\sum_{i=0}^{k_2} \theta_{i,(2)}^2}$, that is when $\theta_{i,(1)} = \theta_{i,(2)}\ \ \forall i$ or, in literal terms, when the smoothing process is the same for both assets. Apart from this special case, this result implies that the interest for these assets will be overestimated since they will present diversification characteristics which are superior to their true ones. We can finally notice that the smoothing process might induce spurious lead-lag correlation between both assets since one can generalize (5) to various lags according to:

$$\operatorname{corr}\left[R^o_{t,(1)};R^o_{t-s,(2)}\right]=\frac{\sum_{i=0}^{\min(k_1,k_2)-s}\theta_{i,(1)}\theta_{i+s,(2)}}{\sqrt{\sum_{i=0}^{k_1}\theta^2_{i,(1)}}\sqrt{\sum_{i=0}^{k_2}\theta^2_{i,(2)}}}\operatorname{corr}\left[R_{t,(1)};R_{t,(2)}\right]\text{ for }0\le s\le\min(k_1,k_2),\qquad(7)$$

and zero elsewhere. Note that (7) generalizes (3) as the latter formula is obtained for the special case where one of the asset is liquid, i.e. $\theta_{0,(j)}=1;\theta_{i,(j)}=0\ \ \forall i>0$.

In Table 3, we present the correction implied by formula (5) for the illiquid assets here studied. In each case, we present the observed contemporaneous correlation on the upper part of the matrix and the corrected correlation on the lower part of the matrix. The ratio between both correlations is equal to 1.2 on average with corrections ranging from 1.05 for the MidWest-West correlation to 1.61 for the National real estate -Venture Capital correlation, a quite significant number.

Fitting AR models in place of MA models

One of the defining features of the smoothing process is the autocorrelation of the observed returns. In the literature, this characteristic has been erroneously interpreted as potentially captured through an autoregressive model, while it is a moving average one (this issue is discussed in further details later). Especially, it has become a standard practice to create unsmoothed returns time-series according to the following steps[6]: (1) estimate an AR model, generally an AR(1) $R^o_t=\rho R^o_{t-1}+u_t$ or an AR(2) $R^o_t=\rho_1R^o_{t-1}+\rho_2R^o_{t-2}+v_t$; (2) create the transformed time-series $Z_t=\left(R^o_t-\hat\rho R^o_{t-1}\right)/\left(1-\hat\rho\right)$ or $Z_t=\left(R^o_t-\hat\rho_1R^o_{t-1}-\hat\rho_2R^o_{t-2}\right)/\left(1-\hat\rho_1-\hat\rho_2\right)$. In theory, this practice is far from being without consequences. In particular, it is well-known that an autoregressive process is far more persistent than a moving average one. While the autocorrelation function of a MA(*k*) process equals zero for a lag superior to *k*, it declines only exponentially for an AR(1) process with the k^{th} autocorrelation coefficient being equal to ρ^k (where ρ is the first-order autocorrelation coefficient). It is our belief that this practice is motivated by the fact that it is easier to estimate an autoregressive model, which can be fitted through a simple OLS regression, than a moving average model which necessitates maximum likelihood or non-linear least-squares.

But easiness seems here detrimental to precision. To the best of our knowledge, such issue is not treated in the standard econometric literature where it is assumed that the form of the process (whether AR, MA or a combination of both) is known or easily identified (through inspection of the autocorrelation function or the use of information criteria). We here propose some asymptotic theoretical and simulation results for various sample sizes. Let us begin with the example of someone who is fitting a AR(1) model on a time-series of observed returns which are smoothed according to a process similar to (1). To keep expressions simple and without loss of generality, we assume that $\mu = 0$. Let ρ_T^* denote the first-order coefficient obtained in the OLS regression with $R_t^o = \rho_T^* R_{t-1}^o + u_t^*$. We have $\rho_T^* = \sum_{t=2}^T R_t^o R_{t-1}^o / \sum_{t=2}^T \left(R_t^o\right)^2$. Asymptotically, this formula simplifies to $\rho^* = E\left(R_t^o R_{t-1}^o\right) / \mathrm{var}\left(R_t^o\right)$. Using the expression for $\mathrm{var}\left(R_t^o\right)$ in (2) and the fact that $E\left(R_t^o R_{t-1}^o\right) = \mathrm{var}\left(R_t\right)\left(\sum_{i=0}^{k-1} \theta_i \theta_{i+1}\right)$, it follows that:

$$\rho^* = \sum_{i=0}^{k-1} \theta_i \theta_{i+1} / \sum_{i=0}^{k} \theta_i^2 . \tag{8}$$

Given that $\theta_i > 0 \ \forall i$, ρ^* is unambiguously positive. Moreover, if the smoothing coefficients are strictly decreasing, $\theta_i \geq \theta_{i+1} \ \forall i$, it is easy to show that ρ^* will necessary be less than unity. In the contrary case, we cannot rule out that $\rho^* > 1$ but one can notice that this is the less plausible the lowest k is. Now, let us imagine that we create a new time-series Z_t with $Z_t = \left(R_t^o - \rho^* R_{t-1}^o\right) / \left(1 - \rho^*\right)$. We deduce that:

$$\frac{\mathrm{var}\left(Z_t\right)}{\mathrm{var}\left(R_t\right)} = \sum_{i=0}^{k} \theta_i^2 \times \frac{\sum_{i=0}^{k} \theta_i^2 + \sum_{i=0}^{k-1} \theta_i \, \theta_{i+1}}{\sum_{i=0}^{k} \theta_i^2 - \sum_{i=0}^{k-1} \theta_i \, \theta_{i+1}} . \tag{9}$$

The ratio in (9) defines whether the true variance of returns is overestimated (if the ratio is superior to 1) or underestimated (if the ratio is inferior to 1). Both outcomes are possible and the final answer is an empirical one even if as we illustrate below, most of the smoothing patterns give rise to a overestimation of the true variance. All we can assert is that overestimation is more probable when k is small. For instance, when $k = 1$, the estimation of the smoothing process through an AR(1) process systematically leads to an overestimation of the variance of returns since $\mathrm{var}\left(Z_t\right) / \mathrm{var}\left(R_t\right) = 1 + \left[\theta_0 \theta_1 \left(1 - \left(\theta_0^2 + \theta_1^2\right)\right)\right] / \left[\theta_0^2 + \theta_1^2 - \theta_0 \theta_1\right] > 1$ where we use the fact that $\theta_0 + \theta_1 = 1$ and $1 > \theta_0^2 + \theta_1^2 > \theta_0 \theta_1$. We can also note that the overestimation is higher when the process is the most smoothed, that is when $\theta_0 = \theta_1 = 0.5$. In this case, we have $\mathrm{var}\left(Z_t\right) / \mathrm{var}\left(R_t\right) = 1.5$. This means that the true variance is overestimated by 50%! This example helps us understand why the usual practice consisting in estimating smoothing processes through

AR models with short lags is misleading. If we fit an AR(1) process, we find that $\rho^* = 0.5$. While the first-order autocorrelation is correctly estimated, the problem is that the AR process leads one to spuriously infer that the underlying process is very persistent since, for instance, it induces that the second-order autocorrelation is 0.25 while it is null in reality. On the contrary, the MA process allows to correctly identify the fact that the process is not persistent.

We can reproduce the same reasoning for the case where the smoothing process is estimated through an AR(2) process and where the corrected time-series is obtained as $Z_t = \left(R_t^o - \rho_1^* R_{t-1}^o - \rho_2^* R_{t-2}^o\right) / \left(1 - \rho_1^* - \rho_2^*\right)$ with:

$$\rho_1^* = \frac{\sum_{i=0}^{k-1} \theta_i \theta_{i+1} \sum_{i=0}^{k} \theta_i^2 - \sum_{i=0}^{k-1} \theta_i \theta_{i+1} \sum_{i=0}^{k-2} \theta_i \theta_{i+2}}{\left(\sum_{i=0}^{k} \theta_i^2\right)^2 - \left(\sum_{i=0}^{k-1} \theta_i \theta_{i+1}\right)^2}, \tag{10a}$$

$$\rho_2^* = \frac{\sum_{i=0}^{k-2} \theta_i \theta_{i+2} \sum_{i=0}^{k} \theta_i^2 - \sum_{i=0}^{k-1} \theta_i \theta_{i+1} \sum_{i=0}^{k-1} \theta_i \theta_{i+1}}{\left(\sum_{i=0}^{k} \theta_i^2\right)^2 - \left(\sum_{i=0}^{k-1} \theta_i \theta_{i+1}\right)^2}, \tag{10b}$$

and

$$\frac{\operatorname{var}(Z_t)}{\operatorname{var}(R_t)} = \frac{\left(1 + \rho_1^{*2} + \rho_2^{*2}\right) \sum_{i=0}^{k} \theta_i^2 - 2\rho_1^* \left(1 - \rho_2^*\right) \sum_{i=0}^{k-1} \theta_i \, \theta_{i+1} - 2\rho_2^* \sum_{i=0}^{k-2} \theta_i \, \theta_{i+2}}{\left(1 - \rho_1^* - \rho_2^*\right)^2}. \tag{11}$$

For finite sample sizes, it is more difficult to derive equivalent formulae. In Table 4A and 4B, we provide simulation evidence for the case of the AR(1) and the AR(2) processes, respectively. We begin by simulating a standard normal variable for fixed sample size, *T*. This time-series is then smoothed according to the various smoothing profiles. We then estimate the autoregressive process and construct a new time-series from this. We finally compute the ratio between the variance of this final time-series relatively to the one of the original time-series (which equals 1 in theory). We call this the overestimation rate. These steps are reproduced 10,000 times for each kind of smoothing process and each form of autoregressive process. From these 10,000 simulations, we draw the sample average value of the ratio and its standard deviation.

In Tables 4A and 4B, we begin by presenting the implied characteristics of the smoothing profiles. We notably see that for each smoothing profile, the smoothness is increasing in the length of the smoothing process, *k*. For the AR(1) process, we see that the various smoothing profiles described by GLM (straight-line, sum-of-year or geometric), the estimation of the smoothing profile through the autoregressive process leads to an overestimation of the variance of the "true" returns in the asymptotic case. For most cases, this overestimation is also observed in finite samples. We also see that overestimation is increasing in sample size and that the overestimation rate is increasing in the length of the smoothing process for the straight-line and sum-of-year cases while it is the contrary for the geometric smoothing pattern. For the AR(2) case, we have quite different results. Cases of underestimation of the variance become far more frequent. The reason is that the persistence of the AR(1) process is partially compensated by the inclusion of the second-order term which is generally negative. Here again, we however observe that overestimation is increasing in the sample size. In the cases of straight-line and sum-of-year smoothing profiles, we observe that the overestimation is still increasing in the length of the smoothing process, *k*, and thus in the smoothness of the process (as shown by the Herfindahl index). In the case of the geometric profiles, the relationship between length of the smoothing process (or smoothness) and overestimation appears non-linear.

In Table 5, we present the correction for illiquidity obtained through AR models. We also compare the values of coefficients obtained in the AR regressions with their theoretical counterparts described above (see equations (8), (10a) and (10b)) where we replace smoothing coefficients θ_i's with their values deduced from MA models (see Table 2). We observe that empirical and theoretical coefficients of the AR models are close in both cases. This tends to validate the framework here adopted. When we compare with Table 2, we clearly see that the corrections for illiquidity obtained through AR models can largely differ from the ones deduced from MA models. Concerning the standard deviation, largest discrepancies are observed for the AR(1) model for real estate indices and for the AR(2) model for venture capital. Concerning the contemporaneous correlation with the factor, with the exception of venture capital, the correction are really at odds with the ones implied by the MA model, both for the AR(1) and the AR(2) models. To sum up this section, we conclude that the choice of estimating the smoothing process through autoregressive process can severely bias the estimation of the variance of the illiquid assets. We strongly recommend the use of MA processes for that purpose.

Combining autoregressive and smoothing processes

The framework developed by GLM is based on the assumption that the true returns are not serially correlated. The assumption behind this hypothesis is the classical Efficient Markets Hypothesis which states that in an efficient market, prices should fluctuate randomly since all relevant information is already incorporated in them, due to the competition between traders, and are thus only perturbated by unexpected news (see e.g. Fama [1970]). A consequence is that the sequence of returns is purely random and, in particular, does not Table serial correlation. The debate on whether the efficient markets hypothesis is verified in practice is largely behind the scope of the present paper but one can observe that except with very high frequency data where microstructure frictions disturb the behavior of prices, the null of no correlation between consecutive returns is largely supported by the data (Campbell, Lo and Mac Kinlay [1997]). In Table 6, we illustrate this result with the monthly and quarterly returns of indices covering the major financial markets in the world (S&P 500, JP Morgan Government Bond World).

Thus it seems reasonable to attribute all the serial correlation observed in data to some form of smoothing behaviour when it comes to analyze the returns of financial assets[7]. We think that this is more questionable when one has to deal with real estate returns or even private equity returns. The reason is that serial correlation is far more frequent for economic variables. In Table 6, we document that point for GDP growth (in real terms) and inflation in the US. We provide additional evidence concerning the autocorrelation structure of real-estate related indices. If no significant dependence is identified for the NAREIT index, we clearly see strong positive autocorrelation coefficient for the housing prices indices. What is interesting in the latter case is that such indices are based on observed transactions and are thus less prone to the kind of smoothing behaviour we might expect for expert-based (appraisers) valuations. It is thus necessary to incorporate the possibility that the serial correlation observed in illiquid assets returns is partly due to a true underlying autoregressive process[8].

We start with the simple case where the true return is driven by an AR(1) process, $R_t = \mu + \rho R_{t-1} + \varepsilon_t$. Combining this dynamics with the smoothing process in (1) and denoting $\eta_t = \theta_0 \varepsilon_t$ and $\vartheta_i = \theta_i / \theta_0$, it follows that observed returns satisfies an ARMA(1,*k*) model $R_t^o = \mu + \rho R_{t-1}^o + \eta_t + \vartheta_1 \eta_{t-1} + \cdots + \vartheta_k \eta_{t-k}$. Even though the dynamics is richer, the approach is as simple as before. First, estimate the parameters through maximum likelihood. Second, deduce smoothing parameters along the same lines as with the MA process: $\hat{\theta}_0 = 1/\left(1 + \hat{\vartheta}_1 + \cdots + \hat{\vartheta}_k\right)$, $\hat{\theta}_i = \hat{\vartheta}_i / \left(1 + \hat{\vartheta}_1 + \cdots + \hat{\vartheta}_k\right)$ $i > 1$. Finally, correct the moments as before but with some variations in formulas to incorporate the AR(1) dynamics[9]:

$$\begin{aligned}
&\operatorname{var}\left(R_t^o\right)=\left(\sum_{i=0}^{k}\sum_{j=0}^{k}\theta_i\theta_j\rho^{|i-j|}\right)\operatorname{var}\left(R_t\right),\\
&\operatorname{cov}(R_t^o,F_t)=\theta_0\operatorname{cov}(R_t,F_t), \qquad (12)\\
&corr(R_t^o,F_t)=\frac{\theta_0}{\sqrt{\sum_{i=0}^{k}\sum_{j=0}^{k}\theta_i\theta_j\rho^{|i-j|}}}corr(R_t,F_t).
\end{aligned}$$

If the process is stationary, that is $|\rho|\leq 1$, it can be shown that the smoothing process again leads to an underestimation of the variance and of the correlation, even when mixed with the AR(1) dynamics. We have estimated this model for the two kinds of alternative assets here investigated[10]. For Venture Capital, the results were not very conclusive. The AR(1) term simply substitutes to the MA(1) one, which becomes not significant different from zero. All in all, a likelihood ratio test indicates that the ARMA(1,3) model does not offer a superior model to the MA(3) model previously chosen. This result tends to confirm that the hypothesis that true returns are not serially dependent seems acceptable for Venture Capital. More satisfactory results are obtained with real estate indexes. For the national index, the estimated ARMA(1,12) leads to a likelihood ratio statistic of 25.04 which is largely superior to the 1% critical value (6.63). The problem is that some MA coefficients become significantly negative, which seems at odds with a form of smoothing behaviour. If we impose the constraints that all MA coefficients should be positive, the likelihood ratio statistic shrinks to 3.96, which is still significant at the 5% level. The Herfindahl index increases to 0.174 from 0.093 (Table 2) which means that the estimated process is less smoothed. This result is not surprising since the generalized model attributes part of the autocorrelation structure to pure autoregressive (economic) phenomenon while the MA model only considers smoothing effects. This implication is also clear if one applies the corrections presented in (12) from which we infer that the corrected standard deviation is 3.31% against 5.47% for the MA model and 1.67% for the original data while the corrected contemporaneous correlation with the 30-yrs mortgage rate is 0.390 against 0.450 for the MA model and 0.222 for the original data.

The generalization of the model should not necessarily be limited to first-order autoregressive dynamics. For instance, we have observed that real estate indexes are characterized by some kind of yearly seasonal pattern with autocorrelation coefficients at multiple values of 4 being larger than their near neighbours. We can thus assume that the process governing the true returns is $R_t=\mu+\rho R_{t-4}+\varepsilon_t$ which, when combined with the smoothing process in (1), leads to an ARMA(4,*k*) process. In practice, we have found that the ARMA(4,3) is offering the most satisfactory results, meaning that the maximum lag of the smoothing process is now equal to 3. For at least 3 among the five regions (National, West and Midwest), the more parsimonious ARMA(4,3) model is superior to the MA models, as shown by lower information criteria[11]. From the ARMA(4,3) model, $R_t^o = \mu + \rho R_{t-4}^o + \eta_t + \vartheta_1 \eta_{t-1} + \vartheta_2 \eta_{t-2} \cdots + \vartheta_3 \eta_{t-3}$, the parameters of the smoothing process are recovered according to $\hat{\theta}_0 = 1/\left(1 + \hat{\vartheta}_1 + \hat{\vartheta}_2 + \hat{\vartheta}_3\right)$ and $\hat{\theta}_i = \hat{\vartheta}_i / \left(1 + \hat{\vartheta}_1 + \hat{\vartheta}_2 + \hat{\vartheta}_3\right)$ for $i > 1$. Given that the lag of the MA is inferior to the lag of the AR, the corrections for the moments are as in the simple MA case, that is (2) and (3), the only difference coming from the estimated values of the smoothing coefficients. In Table 7, we report the correction for the statistics of interest in asset allocation problems. We observe that these statistics are larger than the one directly inferred from data but they are smaller than the one obtained with the MA model. Again, this reflects the fact that with autoregressive dynamics, one is putting less emphasis on smoothing, as clearly shown by the comparisons of Herfindahl indexes in Table 2 and Table 7.

This section has clearly shown that for some alternative assets and notably for real estate, it is necessary to go beyond the hypothesis of a pure random process for the true returns and to jointly estimate smoothing and autoregressive processes. If one is to ignore this aspect, the danger is to overvalue the smoothing pattern of the returns and thus to infer corrections for statistics which might be too disadvantageous for illiquid assets. One limitation is that generalization to ARMA processes with higher lags for the autoregressive part leads to more complex theoretical corrections. It is our belief that for most alternative assets, low order processes are sufficient. If necessary, a simpler procedure, which might not be too inefficient for large enough sample size, is as follows : (i) initialize the process with theoretical values for $t \le 0$ such as $\mu \times \left(1 - \sum_{l=1}^{p} \rho_l\right)^{-1}$ where μ is the estimated average of the observed return and the ρ_l's are the coefficients and p the maximum lag of the AR part of the process; (ii) deduce the corrected time-series R_t^* recursively according to $R_t^* = \left(R_t^o - \theta_1 R_{t-1}^* - \cdots - \theta_k R_{t-k}^*\right) / \theta_0$ for $t = 1, \ldots, T$.

6. Conclusion: an example of implications for asset allocation

In this paper, we have extended previous researches on the implications of illiquidity and returns smoothing for asset allocation. Starting from the model recently introduced in GLM, we have offered numerous new theoretical and empirical results which are important in practice. First, we have extended previous results for the case of various illiquid assets. Second, we have provided simulation and empirical evidences that the standard practice to unsmooth the time-series through autoregressive processes can be misleading. Third, we have shown that, for some alternative assets, it is relevant and even necessary to incorporate jointly autoregressive dynamics along the smoothing process.

To illustrate further the implications of our results, we end with a typical asset allocation exercise. It consists in establishing optimal portfolios mixing two alternative assets, real estate (national index) and Venture Capital, with a standard asset, the Nasdaq. We aim to compare portfolios composition when one considers raw statistics, as drawn from original time-series of returns, and corrected statistics. For real estate, the correction is based on the ARMA(4,3) model presented in Table 7 while for Venture Capital the correction is based on the MA(3) model presented in Table 2. The correction leads to higher standard deviation for both assets. Concerning the correlation matrix, two noteworthy features emerge. First, the correlation between both illiquid assets is almost unchanged. This result is very specific to this empirical application and is due to the fact that the smoothing coefficients are very similar for both time-series (see above). Second, if the correlation between real estate and Nasdaq increases in absolute value as a result of the de-smoothing, it becomes more negative since the original sign is negative. Thus, the attempt to remove smoothed characteristic of the time-series conveys higher diversification properties for the illiquid asset. For Venture Capital, we observe the more intuitive result of diminished diversification properties.

We then solve the standard Markowitz quadratic problem, with no short sales and budget constraints, for various levels of risk loving λ. All in all, the results, which are summarized in Table 8, clearly illustrate the implications of the correction of the statistics as the optimal portfolios are characterized by a sharp fall in Venture Capital proportion, which is the most affected by the correction, in favour of real estate and, above all, the Nasdaq. Correcting for illiquidity is clearly a key step in asset allocations problems which involve alternative assets.

NOTES

[1] Another way to consider the implications of illiquidity is in terms of periodic release of the accumulated value of the asset, which is tantamount to the case of the stale pricing problem analyzed in details in Scholes and Williams [1977] and Lo and McKinlay [1990] (see Asness, Kail and Liew [2001] for a discussion in the case of hedge funds). Stale pricing has huge implications on correlation which is downward biased but none on volatility (in fact volatility can even be upward biased if the mean return is different from zero) and above all does not lead to any autocorrelation of observed returns, thus being inconsistent with empirical facts for alternative assets.

[2] For an introduction and a discussion of the NCREIF indexes, see Fisher [2005].

[3] Details of the estimation are available in the working paper version of the article, where we compare the moving average method with two alternative methods, one based on a factor model and the other which relies on lower frequency data.

[4] More generally, the values of the Herfindahl index found here are typically lower than the ones put forward in GLM, confirming our conjecture that hedge funds are the less illiquid of illiquid assets.

[5] The factor is here taken as an example. We could obviously analyze the implications of the smoothing pattern with any variable under the maintained hypothesis that this variable is not itself subject to a smoothing behaviour. We relax this last hypothesis later.

[6] See, among others, Budhrajha and de Figueiredo [2004] or Sherer [2004].

[7] Note however that GLM suggest three other reasons why returns might be serially correlated without any smoothing process. The first one is the possibility of time-varying expected returns. The two others are more specific to the hedge funds and are time-varing leverage and high water mark incentive fees. They conclude that none is able to justify the serial correlation observed in hedge funds returns.

[8] It is beyond the scope of this paper to analyze the sources of this serial correlation in economic variables. We can however suggest several explanations: time-varying expected economic growth, physical persistent phenomenon, extrapolative expectations. In the case of real estate, this last explanation is deemed to be particular meaningful since there is large evidence that household expectations Table such a pattern concerning housing prices (see, for instance, Case, Quigley and Shiller [2004]). Note however that even for indices which are based on transaction prices, some form of smoothing might appear due to temporal aggregation; see Geltner [1993].

[9] Obviously, one can generalize the formula to the case where the factor is itself smoothed and / or autocorrelated.

[10] All the estimation details are available upon request.

[11] One should notice that the ARMA(4,3) model for quarterly data implies an AR(1) model at an annual frequency.

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Table 1. Descriptive statistics.

	Real estate					Venture
	National	**East**	**South**	**West**	**Midwest**	**Capital**
Time period	1978Q1-2004Q3	1978Q1-2004Q3	1978Q1-2004Q3	1978Q1-2004Q3	1978Q1-2004Q3	1986Q3-2003Q4
Number of observations	108	108	108	108	108	70
Average	2.32	2.69	1.99	2.42	2.06	3.88
Standard deviation	1.67	2.32	1.60	2.05	1.48	9.86
Correlation with the factor ([a])	0.222	0.344	0.102	0.189	0.035	0.566
Autocorrelation coefficients						
1	0.680 ***	0.575 ***	0.578 ***	0.555 ***	0.366 ***	0.541 ***
2	0.675 ***	0.594 ***	0.539 ***	0.584 ***	0.480 ***	0.467 ***
3	0.588 ***	0.500 ***	0.440 ***	0.498 ***	0.323 ***	0.289 **
4	0.697 ***	0.515 ***	0.449 ***	0.712 ***	0.635 ***	0.057
5	0.455 ***	0.502 ***	0.319 ***	0.360 ***	0.207 **	-0.032
6	0.409 ***	0.321 ***	0.327 ***	0.365 ***	0.283 ***	-0.069
7	0.368 ***	0.284 ***	0.278 ***	0.329 ***	0.182 *	-0.007
8	0.422 ***	0.265 ***	0.285 ***	0.451 ***	0.327 ***	-0.051
9	0.239 **	0.172 *	0.242 **	0.196 **	0.031	-0.084
10	0.223 **	0.191 **	0.160 *	0.225 **	0.109	-0.026
11	0.174 *	0.127	0.139	0.153	0.060	-0.234 *
12	0.190 **	0.109	0.147	0.205 **	0.110	-0.168

Notes.
Mean and standard deviation are expressed as % per quarter.
([a]) The factor is assumed to be the 30-yrs mortgage rate for the real estate returns and the Nasdaq for the venture capital returns.
***, **, * denotes rejection of the null hypothesis of a zero autocorrelation coefficient at the 1%, 5% and 10% significance level, respectively. It is assumed that the autocorrelation coefficient is distributed as a standard normal variable with mean zero and variance T for T being the number of observations.

Table 2. Smoothing behaviour inferred from MA models.

	Real estate					Venture
	National	East	South	West	Midwest	Capital
			Smoothing coefficients			
θ_0	0.151	0.241	0.212	0.212	0.295	0.454
θ_1	0.058	0.070	0.047	0.087	0.057	0.164
θ_2	0.088	0.107	0.095	0.082	0.098	0.210
θ_3	0.051	0.043	0.085	0.081	0.058	0.172
θ_4	0.146	0.128	0.123	0.189	0.214	--
θ_5	0.076	0.146	0.036	0.075	0.055	--
θ_6	0.054	0.039	0.135	0.033	0.074	--
θ_7	0.071	0.049	0.039	0.028	0.049	--
θ_8	0.101	0.073	0.166	0.132	0.101	--
θ_9	0.055	0.015	0.061	0.008	--	--
θ_{10}	0.063	0.090	--	0.073	--	--
θ_{11}	0.026	--	--	--	--	--
θ_{12}	0.061	--	--	--	--	--
Herfindahl index $\sum_{i=0}^{k} \theta_i^2$	0.093	0.131	0.131	0.132	0.170	0.307
			Standard deviation (% per quarter)			
Observed	1.67	2.32	1.60	2.05	1.48	9.86
Corrected	5.47	6.41	4.41	5.63	3.60	17.80
			Contemporaneous correlation with the factor			
Observed	0.222	0.344	0.102	0.189	0.035	0.566
Corrected	0.450	0.518	0.174	0.324	0.049	0.690

Notes.
The upper part shows the smoothing coefficients implied by the estimation of MA models where the maximum lag *k* is fixed according to the inspection of the autocorrelation function of each time-series. The lower part presents the corrected statistics, based on formulas (2) and (3) given in the body part of the text.

Table 3. Correlation between illiquid assets.

	National	East	South	West	Midwest	Venture Capital
National	---	0.889	0.801	0.926	0.806	0.035
East	0.972	---	0.619	0.714	0.706	-0.011
South	0.900	0.743	---	0.691	0.574	-0.032
West	0.989	0.761	0.772	---	0.647	0.113
Midwest	0.901	0.777	0.629	0.682	---	0.000
Venture Capital	0.057	-0.015	-0.046	0.161	0.000	---

Notes.
The Table reports the contemporaneous correlation coefficients between illiquid assets. The numbers on the upper part of the matrix are based on observed returns while the ones on the lower part are corrected for the smoothing process where correction is based on formula (5) and MA-implied smoothing coefficients.

Table 4A. Overestimation rate of the true variance with an AR(1) for various smoothing profiles.

Parameters							**Implied characteristics**			**Overestimation rate**					
k	θ_0 **(%)**	θ_1 **(%)**	θ_2 **(%)**	θ_3 **(%)**	θ_4 **(%)**	θ_5 **(%)**	$\sum_{i=0}^{k}\theta_i^2$	$\sum_{i=0}^{k-1}\theta_i\theta_{i+1}$	ρ^*	$T=\infty$	$T=25$	$T=50$	$T=100$	$T=1000$	$T=10000$
							Straight-line smoothing								
1	50.0	50.0	--	--	--	--	0.500	0.250	0.500	1.500	1.474	1.475	1.482	1.501	1.499
											(1.09)	(0.60)	(0.43)	(0.13)	(0.04)
2	33.3	33.3	33.3	--	--	--	0.333	0.222	0.667	1.667	1.546	1.647	1.657	1.655	1.667
											(1.45)	(0.95)	(0.60)	(0.18)	(0.06)
3	25.0	25.0	25.0	25.0	--	--	0.250	0.188	0.750	1.750	1.666	1.624	1.736	1.738	1.754
											(2.09)	(1.12)	(0.74)	(0.23)	(0.07)
4	20.0	20.0	20.0	20.0	20.0	--	0.200	0.160	0.800	1.800	1.634	1.667	1.704	1.797	1.799
											(2.25)	(1.19)	(0.86)	(0.29)	(0.09)
5	16.7	16.7	16.7	16.7	16.7	16.7	0.167	0.139	0.833	1.833	1.529	1.750	1.761	1.854	1.831
											(2.09)	(1.76)	(1.02)	(0.31)	(0.10)
							Sum-of-year smoothing								
1	66.7	33.3	--	--	--	--	0.556	0.222	0.400	1.296	1.237	1.274	1.284	1.294	1.296
											(0.87)	(0.51)	(0.35)	(0.10)	(0.03)
2	50.0	33.3	16.7	--			0.389	0.222	0.571	1.425	1.266	1.367	1.377	1.427	1.423
											(1.00)	(0.70)	(0.46)	(0.14)	(0.05)
3	40.0	30.0	20.0	10.0	--	--	0.300	0.200	0.667	1.500	1.331	1.433	1.440	1.493	1.501
											(1.38)	(0.84)	(0.53)	(0.18)	(0.06)
4	33.3	26.7	20.0	13.3	6.7	--	0.244	0.178	0.728	1.551	1.376	1.411	1.538	1.548	1.553
											(1.86)	(0.96)	(0.68)	(0.21)	(0.07)
5	28.6	23.8	19.0	14.3	9.5	4.8	0.206	0.159	0.769	1.578	1.444	1.578	1.543	1.575	1.579
											(1.82)	(1.38)	(0.71)	(0.24)	(0.07)
							Geometric smoothing ($\delta = 0.25$**)**								
1	80.0	20.0	--	--	--	--	0.680	0.160	0.235	1.098	1.008	1.092	1.090	1.098	1.099
											(0.57)	(0.42)	(0.28)	(0.08)	(0.03)
2	76.2	19.0	4.8	--	--	--	0.619	0.154	0.249	1.029	1.013	0.999	1.025	1.029	1.028
											(0.68)	(0.40)	(0.29)	(0.08)	(0.03)
3	75.3	18.8	4.7	1.2	--	--	0.605	0.151	0.250	1.007	0.994	0.994	1.000	1.009	1.008
											(0.67)	(0.41)	(0.27)	(0.08)	(0.03)
4	75.0	18.8	4.7	1.2	0.3	--	0.600	0.150	0.251	1.002	0.970	0.987	1.006	0.999	1.001
											(0.64)	(0.41)	(0.29)	(0.08)	(0.03)
5	74.9	18.8	4.7	1.2	0.3	0.1	0.599	0.150	0.251	1.000	0.960	0.996	0.991	1.003	1.000
											(0.66)	(0.42)	(0.27)	(0.08)	(0.03)

Notes.

Straight-line smoothing is characterized by $\theta_i = (k+1)^{-1}$, Sum-of-Years smoothing by $\theta_i = (k+1-i)/((k+1)(k+2)/2)$ and geometric smoothing by $\theta_i = \delta^i(1-\delta)(1-\delta^{k+1})$. $T=\infty$ corresponds to the asymptotic values described in equation (8). For other values of T, we present both the average value and in parentheses the standard deviation obtained in 10,000 simulations, the true returns being assumed to be drawn from a standard normal distribution.

Table 4B. Overestimation rate of the true variance with an AR(2) for various smoothing profiles.

	Parameters						**Implied characteristics**					**Overestimation rate**					
k	θ_0 **(%)**	θ_1 **(%)**	θ_2 **(%)**	θ_3 **(%)**	θ_4 **(%)**	θ_5 **(%)**	$\sum_{i=0}^{k}\theta_i^2$	$\sum_{i=0}^{k-1}\theta_i\theta_{i+1}$	$\sum_{i=0}^{k-2}\theta_i\theta_{i+2}$	ρ_1^*	ρ_2^*	$T = \infty$	$T = 25$	$T = 50$	$T = 100$	$T = 1000$	$T = 10000$
									Straight-line smoothing								
1	50.0	50.0	--	--	--	--	0.500	0.250	0.000	0.67	-0.33	0.75	0.63	0.71	0.72	0.75	0.75
													(0.68)	(0.4)	(0.26)	(0.08)	(0.03)
2	33.3	33.3	33.3	--	--	--	0.333	0.222	0.111	0.80	-0.20	1.11	0.94	0.98	1.04	1.11	1.11
													(1.07)	(0.63)	(0.45)	(0.14)	(0.04)
3	25.0	25.0	25.0	25.0	--	--	0.250	0.188	0.125	0.86	-0.14	1.31	1.05	1.13	1.25	1.30	1.31
													(1.58)	(0.84)	(0.59)	(0.17)	(0.05)
4	20.0	20.0	20.0	20.0	20.0	--	0.200	0.160	0.120	0.89	-0.11	1.44	1.02	1.28	1.36	1.42	1.44
													(1.68)	(1.1)	(0.69)	(0.21)	(0.07)
5	16.7	16.7	16.7	16.7	16.7	16.7	0.167	0.139	0.111	0.91	-0.09	1.53	1.14	1.36	1.40	1.52	1.52
													(2.04)	(1.38)	(0.77)	(0.25)	(0.08)
									Sum-of-year smoothing								
1	66.7	33.3	--	--	--	--	0.556	0.222	0.000	0.48	-0.19	0.88	0.78	0.84	0.84	0.88	0.88
													(1.04)	(0.44)	(0.29)	(0.09)	(0.03)
2	50.0	33.3	16.7	--			0.389	0.222	0.084	0.67	-0.17	1.02	0.94	0.96	1.00	1.01	1.02
													(1.16)	(0.57)	(0.4)	(0.11)	(0.04)
3	40.0	30.0	20.0	10.0	--	--	0.300	0.200	0.110	0.76	-0.14	1.13	0.91	1.01	1.09	1.12	1.13
													(0.96)	(0.65)	(0.44)	(0.14)	(0.04)
4	33.3	26.7	20.0	13.3	6.7	--	0.244	0.178	0.116	0.82	-0.12	1.22	0.95	1.10	1.15	1.21	1.21
													(1.3)	(0.75)	(0.53)	(0.16)	(0.05)
5	28.6	23.8	19.0	14.3	9.5	4.8	0.206	0.159	0.113	0.85	-0.10	1.28	0.97	1.11	1.22	1.28	1.28
													(1.29)	(0.93)	(0.61)	(0.19)	(0.06)
									Geometric smoothing ($\delta = 0.25$ **)**								
1	80.0	20.0	--	--	--	--	0.680	0.160	0.000	0.25	-0.06	0.98	0.85	0.93	0.96	0.97	0.98
													(0.81)	(0.45)	(0.3)	(0.1)	(0.03)
2	76.2	19.0	4.8	--	--	--	0.619	0.154	0.037	0.25	0.00	1.02	0.89	0.94	0.99	1.02	1.02
													(0.79)	(0.52)	(0.33)	(0.11)	(0.03)
3	75.3	18.8	4.7	1.2	--	--	0.605	0.151	0.038	0.25	0.00	1.01	0.86	0.98	0.97	1.01	1.01
													(0.73)	(0.51)	(0.33)	(0.1)	(0.03)
4	75.0	18.8	4.7	1.2	0.3	--	0.600	0.150	0.038	0.25	0.00	1.00	0.89	0.96	0.98	1.00	1.00
													(0.85)	(0.56)	(0.34)	(0.1)	(0.03)
5	74.9	18.8	4.7	1.2	0.3	0.1	0.599	0.150	0.038	0.25	0.00	1.00	0.85	0.94	0.99	1.00	1.00
									*				(0.82)	(0.53)	(0.36)	(0.1)	(0.03)

Notes.

Straight-line smoothing is characterized by $\theta_i = (k+1)^{-1}$, Sum-of-Years smoothing by $\theta_i = (k+1-i)/((k+1)(k+2)/2)$ and geometric smoothing by $\theta_i = \delta^i(1-\delta)(1-\delta^{k+1})$. $T = \infty$ corresponds to the asymptotic values described in equations (10a and 10b). For other values of *T*, we present both the average value and in parentheses the standard deviation obtained in 10,000 simulations, the true returns being assumed to be drawn from a standard normal distribution.

Table 5. Corrected statistics with autoregressive models.

		Real estate					
		National	East	South	West	Midwest	Venture Capital
				AR(1) model			
First-order coefficient :	observed	0.687	0.576	0.595	0.559	0.368	0.535
	theoretical	0.689	0.507	0.489	0.533	0.381	0.472
Corrected standard deviation		3.90	4.49	3.17	3.86	2.19	17.93
Corrected correlation with factor		0.194	0.426	-0.023	0.115	0.018	0.622
				AR(2) model			
First-order coefficient :	observed	0.412	0.347	0.409	0.334	0.221	0.403
	theoretical	0.366	0.315	0.241	0.358	0.231	0.364
Second-order coefficient :	observed	0.399	0.392	0.312	0.402	0.401	0.240
	theoretical	0.469	0.379	0.507	0.328	0.395	0.231
Corrected standard deviation		5.93	6.71	4.39	5.91	3.35	22.81
Corrected correlation with factor		0.198	0.380	-0.027	0.167	-0.027	0.671

Notes.
The Table presents the results of the application of autoregressive (AR) models to unsmooth the real estate and Venture Capital returns. We present coefficients associated with AR models, both for their empirical estimated values and their theoretical values given in the body part of the text when one assumes that the smoothing process is similar to eq. (1) and where we replace smoothing coefficients θ_i's with their empirical values deduced from MA models (see Table 2).

Table 6. Autocorrelation structure for various economic and financial variables.

Variable	Frequency	Time period	Number of observations	Autocorrelation coefficients Lag 1	Lag 2	Lag 3	Lag 4
			Financial variables				
S&P	Monthly	1965:1-2005:6	486	0.005	-0.037	0.017	-0.026
	Quarterly	1965:1-2005:1	161	0.045	-0.053	-0.0247	0.015
JP Morgan World government bond index	Monthly	1986:1-2005:6	234	0.092	-0.133 **	-0.002	0.035
	Quarterly	1986:2-2005:1	76	-0.031	-0.091	0.064	-0.079
NAREIT	Monthly	1972:2-2005:6	401	0.060	0.011	0.021	0.051
	Quarterly	1972:2-2005:1	132	0.092	0.000	0.034	0.103
			Economic variables				
US GDP	Quarterly	1951:1-2005:1	217	0.321 ***	0.157 **	-0.023	-0.104
	Yearly	1951-2004	54	0.032	-0.072	-0.212	0.054
US CPI	Quarterly	1951:1-2005:1	217	0.761 ***	0.647 ***	0.734 ***	0.648 ***
	Yearly	1951-2004	54	0.801 ***	0.545 ***	0.410 ***	0.396 ***
			Real estate variables				
US Housing prices (OFHEO)	Quarterly	1975:1-2005:1	120	0.568 ***	0.523 ***	0.566 ***	0.530 ***
	Yearly	1976-2004	29	0.746 ***	0.517 ***	0.174	-0.105
France Housing prices (INSEE-Notaires)	Quarterly	1992:1-2005:1	52	0.625 ***	0.580 ***	0.635 ***	0.760 ***
	Yearly	1992-2004	13	0.839 ***	0.643 **	0.520 *	0.491 *

Notes.
All variables are transformed as one-period returns.
***, **, * denotes rejection of the null hypothesis of a zero autocorrelation coefficient at the 1%, 5% and 10% significance level, respectively. It is assumed that the autocorrelation coefficient is distributed as a standard normal variable with mean zero and variance T for T being the number of observations.

Table 7. Smoothing behaviour inferred from ARMA(4,3) models for real estate indexes.

	National	East	South	West	Midwest
	Implied smoothing coefficients				
θ_0	0.441	0.561	0.425	0.392	0.589
θ_1	0.177	0.116	0.191	0.171	0.120
θ_2	0.228	0.240	0.197	0.237	0.184
θ_3	0.155	0.083	0.187	0.200	0.107
Herfindahl index $\sum_{i=0}^{k} \theta_i^2$	0.301	0.392	0.291	0.279	0.407
	Corrected statistics				
Standard deviation	3.04	3.71	2.96	3.87	2.32
Correlation with factor	0.277	0.384	0.129	0.255	0.038

Notes.
The upper part of the Table reports the smoothing coefficients implied by the estimation of ARMA(*4,3*) process on observed returns. The corrected statistics are obtained according to eq. (2) for standard deviation and eq. (3) for the contemporaneous correlation with the factor (here, the 30-yrs mortgage rate). The raw ("observed") statistics are given in Table 2.

Table 8. An asset allocation exercise.

		Real estate national index	Venture Capital	Nasdaq
		Statistics		
Average		1.71	3.88	3.35
Standard deviation	Raw	1.56	9.86	14.47
	Corrected	2.85	17.80	14.47
Correlation matrix		1	0.106	-0.069
(raw statistics on upper part ;		0.106	1	0.566
corrected on lower part)		-0.085	0.690	1
		Asset allocation: composition of optimal portfolios		
		High risk aversion ($\lambda = 0$)		
	Raw	98.1%	0.0%	1.9%
	Corrected	94.8%	0.0%	5.2%
		Moderate risk aversion ($\lambda = 0.5$)		
	Raw	42.9%	57.1%	0%
	Corrected	75.0%	8.5%	16.5%
		Low risk aversion ($\lambda = 1$)		
	Raw	0.0%	100.0%	0.0%
	Corrected	54.4%	21.2%	24.4%

Notes.
The Table presents the results of an asset allocation exercise involving two illiquid assets, real estate and Venture Capital, and a liquid one, the Nasdaq. The composition of portfolios is obtained through the standard Markowitz-quadratic problem of with short sales and budget constraints. The optimisation is done for raw and corrected statistics. Raw statistics are calculated over the period 1986Q3-2003Q4. For real estate, the correction is based on an ARMA(4,3) model. For Venture Capital, it is based on a MA(3) model.