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TERM STRUCTURE MODELS OF COMMODITY PRICES

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ABSTRACT. This review article describes the main contributions in the literature on term structure models of commodity prices. A first section is devoted to the theoretical analysis of the term structure. It confines itself primarily to the traditional theories of commodity prices and to their explanation of the relationship between spot and futures prices. The normal backwardation and storage theories are however a bit limited when the whole term structure is taken into account. As a result, there is a need for an extension of the analysis for long-term horizon, which constitutes the second point of the section. Finally, a dynamic analysis of the term structure is presented. Section two is centred on term structure models of commodity prices. The presentation shows that these models differ on the nature and the number of factors used to describe uncertainty. Four different factors are generally used: the spot price, the convenience yield, the interest rate, and the long-term price. Section three reviews the main empirical results obtained with term structure models. First of all, simulations highlight the influence of the assumptions concerning the stochastic process retained for the state variables and the number of state variables. Then, the method usually employed for the estimation of the parameters is explained. Lastly, the models' performances, namely their ability to reproduce the term structure of commodity prices, are presented. Section four exposes the two main applications of term structure models: hedging and valuation. Section five resumes the broad trends in the literature on commodity pricing during the 1990s and early 2000s, and proposes futures directions for research.

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ABSTRACT. This review article describes the main contributions in the literature on term structure models of commodity prices. A first section is devoted to the theoretical analysis of the term structure. It confines itself primarily to the traditional theories of commodity prices and to their explanation of the relationship between spot and futures prices. The normal backwardation and storage theories are however a bit limited when the whole term structure is taken into account. As a result, there is a need for an extension of the analysis for long-term horizon, which constitutes the second point of the section. Finally, a dynamic analysis of the term structure is presented. Section two is centred on term structure models of commodity prices. The presentation shows that these models differ on the nature and the number of factors used to describe uncertainty. Four different factors are generally used: the spot price, the convenience yield, the interest rate, and the long-term price. Section three reviews the main empirical results obtained with term structure models. First of all, simulations highlight the influence of the assumptions concerning the stochastic process retained for the state variables and the number of state variables. Then, the method usually employed for the estimation of the parameters is explained. Lastly, the models' performances, namely their ability to reproduce the term structure of commodity prices, are presented. Section four exposes the two main applications of term structure models: hedging and valuation. Section five resumes the broad trends in the literature on commodity pricing during the 1990s and early 2000s, and proposes futures directions for research.

Key words : term structure – commodity – futures prices – term structure models – crude oil – storage theory – normal backwardation theory – hedging – valuation – Samuelson effect – real options – Kalman filter

INTRODUCTION

This article reviews the literature on term structure of commodity prices with particular reference to recent developments on term structure models and their applications. The term structure is defined as the relationship between the spot price and the futures prices for any delivery date. It provides useful information for hedging or investment decision, because it synthesizes the information available in the market and the operators' expectations concerning the future. This information is very useful for management purposes: it can be used to hedge exposures on the physical market, to adjust the stocks level or the production rate. It can also be used to undertake arbitrage transactions, to evaluate derivatives instruments based on futures contracts, etc. In many commodity markets, the concept of term structure becomes important, because the contracts' maturity increases as the markets ripen.

In the American crude oil market, this ripening process has gone very far because since 1999, there are futures contracts for maturities as far as seven years. Thus, this market is today the most developed commodity futures market, considering the volume and the maturity of the transactions. It provides publicly available prices – namely potentially informative and costless signals – whereas in most commodities markets, the only information for far maturities is private and given by forward prices. The introduction of these long term futures contracts authorizes empirical studies on the crude oil prices' curves that were only possible before with forward prices, whose informational content is not necessarily reliable or workable (forward contracts are not standardized, and the prices reporting mechanism does not force the operators to disclose their transactions prices).

This specificity of the crude oil futures market explains why most of the examples used in this article refer themselves to this commodity. Moreover, even if there are now, in all the commodity markets, miscellaneous derivatives markets and hedging instruments (options, swaps, etc), we concentrate on the futures contracts, in order to focus on the relationship between the physical and the paper markets. However, the analysis presented can be extended to other derivatives products and markets, because the pricing of complicated instruments can always be reduced to the determination of the term structure of futures prices.

The reader can find reviews of the literature on commodity markets that are more general. A recent and extensive review can be found in Carter (1999). Gray and Rutledge (1971) also propose a well-known review on the evolutionary aspects of futures markets, on inter-temporal price relationships, on concepts of hedging, on the stochastic nature of price fluctuations, etc. Other reviews were also done by Goss and Yamey (1978), Tomek and Robinson (1977), Kamara (1982), Blank (1989), Malliaris (1997), etc.

This survey proceeds as follows. A first section is devoted to the theoretical analysis of the term structure. Section two is centred on term structure models of commodity prices. Section three analyses the ability of the models to describe the prices curve empirically observed. Section four reviews the two main applications of term structure models: hedging and valuation. Section five concludes.

SECTION 1. THEORETICAL ANALYSIS OF THE TERM STRUCTURE

This section primarily confines itself to the traditional theories of commodity prices and to their explanation of the relationship between spot and futures prices. The normal backwardation and the storage theories are however a bit limited when the whole term structure is taken into account. As a result, there is a need for an extension of the analysis for long-term horizon, which constitutes the second point of the section. Finally, a dynamic analysis of the term structure is presented.

1.1. Traditional theories and the explanation of the relationship between spot and futures prices

The normal backwardation and the storage theories are traditionally used to explain the relationship between spot and futures prices in commodity markets. Whereas the theory of normal backwardation is centred on the analysis of hedging positions and on the function of transferring the risk provided by the futures market, the storage theory proposes an explanation based on the storage costs. More precisely, the different determinants of the futures prices, in this context, are the spot price, the convenience yield, and the storage cost. The latter includes the pure storage cost and the financing cost.

Keynes introduced the theory of normal backwardation in 1930. Briefly summarized, its central argument is the following: in normal conditions, the commodity market is characterised by a forward price situated below the spot price:

> *" ...in normal conditions the spot price exceeds the forward price i.e. there is backwardation. In other words, the normal supply price on the spot includes remuneration for the risk of price fluctuation during the period of production, whilst the forward price excludes this."»*[1]

The relationship linking these two prices is due to the relative importance of short and long hedging positions in the futures market. The first assumption of the theory is that short hedging represents a lower volume than long hedging. Consequently, there is a need for speculators to compensate for this market unbalance. In order to motivate the speculators' intervention, there must be a difference between the futures price and the spot price expected at the contract's delivery date. This is the second assumption of the theory. The presence of a positive risk premium associated with the expected spot price explains the difference between the spot and the futures prices. This premium remunerates the speculators for the risks they undertake in their activity.

Until now, the theory of normal backwardation was never truly validated nor rejected. Dusak (1973), Bodie and Rosansky (1980), Richard and Sudaresan (1981), Bessembinder (1993) use either static or inter-temporal capital asset pricing model to examine the futures risk premium and they obtained contradictory results. The usual critics raised against the theory of normal backwardation sustain that whenever it exists, there a few chances that the premium is positive and constant. Indeed, the net position of the hedgers in the commodity futures markets is not always a short position. Moreover, the risk aversion of the participants can change with time. As a result, the empirical tests carried out in order to validate the theory are contradictory: for the same futures market and on different periods, they can conclude either that there is normal backwardation or conversely that there is "normal contango".

The storage theory relies on the reasons explaining the holding of physical stocks to understand the relationships between spot and futures prices in commodity markets. The analysis of the arbitrage operations between the physical and the futures markets makes it possible to understand the mechanisms causing contango and backwardation. It also shows that the basis evolves differently when it is positive or negative. Indeed, contango is limited to the storage costs between the current date and the contract's expiration. Such a limit does not exist for backwardation.

When physical stocks are invoked to explain the relationship between spot price and futures price, interpreting backwardation becomes tricky. If the futures price corresponds to the spot price increased by positive storage costs, how can we explain that sometimes, the futures price is below the spot price? The concept of convenience yield, introduced by Kaldor in 1939, brings an answer to this question. The convenience yield can be briefly defined as the implicit revenue associated with stock holding. The possession of inventories indeed avoids the costs of frequent supply orders and spares the waiting time associated to deliveries:

[1] Keynes, 1930.

"In normal circumstances, stocks of all goods possess a yield, measured in terms of themselves, and this yield which is a compensation to the holder of stocks, must be deducted from carrying costs proper in calculating net carrying cost. The latter can, therefore, be negative or positive."[2]

Moreover, as Brennan (1958) stated it, inventories give the possibility to take advantage of sudden and unexpected rises in the demand:

"The convenience yield is attributed to the advantage (in terms of less delay and lower costs) of being able to keep regular customers satisfied or of being able to take advantage of a rise in demand and price without resorting to a revision of the production schedule."

These definitions show that the convenience yield is high when inventories are rare, because stock holding is all the more appreciated, as the stock level is low. Conversely, the convenience yield is low when stocks are abundant. Moreover, the convenience yield is positively correlated with the spot price, which is also high when there is a shortage of stocks, and conversely.

An important part of the research on storage theory was devoted to the definition and the analysis of the convenience yield. This concept is central for the analysis of the term structure of commodity prices. In the context of financial markets, the convenience yield corresponds to the coupon linked with the bonds or to the dividends given by a stock portfolio.

The storage theory constitutes the main basis for the elaboration of term structure models of commodity prices. Indeed, it brings useful conclusions to construct such a model. First, the relationship between spot and future prices allows the identification of at least three variables influencing the future price: the spot price, the convenience yield net of storage costs, and the interest rate, which is implicitly included in the financing costs. Second, convenience yield and spot price are positively correlated: both of them are an inverse function of stocks level. Third, the examination of arbitrage relationships between physical and paper markets shows that the basis has an asymmetrical behavior: in contango, its level is limited to storage costs. This is not the case in backwardation. Furthermore, the basis is stable in contango, and volatile in backwardation, since in this situation stocks cannot absorb price fluctuations. This asymmetry has implications on the dynamic of convenience yield that were exploited.

1.2. Extension of the analysis to long-term horizon

The most important developments concerning the traditional theories of commodity prices were introduced between 1930 and 1960. At that time, the transactions' horizon on futures markets was rarely longer than one year. Therefore, the analysis was originally conceived for the short term, and it has to be adapted in order to enable a long-term analysis.

The Keynesian analysis can be extended rather simply. When the whole prices curve is taken into account, the eventual simultaneous presence of contango and backwardations along the curve can be explained by a surplus in the supply or in the demand of futures contracts for specific maturities. In order to palliate these unbalances, provided they accept to take a position in the futures market that

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[2] Kaldor, 1939.

compensate for the net position of the others operators, a risk premium is offered to the speculators. In the way of the preferred habitat theory developed for interest rates (Modigliani and Sutch, (1966)), the term structure of commodity prices is then regarded as a succession of segments having different maturities. Market participants select their segment in step with their economic needs.

Such an extension of the Keynesian analysis amounts to removing the assumptions concerning the sign and the level of the risk premium. In this context, there are several unbalances between the contract's supply and demand. Each segment of the prices curve is supposed satisfying a specific economic need. Therefore, all the categories of operators do not necessarily intervene on all the maturities. The level of the premium they are willing to pay and the sense of market's unbalance can be different for each segment. Thus, the risk premium is a function of the maturity. Moreover, in order to take into account the eventual distortions of the prices curves, this premium must vary with the period, as the operators' expectations and risk aversion change.

As far as the storage theory is concerned, it takes at first into account the existence of a term structure of contangos and backwardations. In this context, such a phenomenon is due to the seasonality of the supply or the demand for the commodity. For example, the coexistence of a short-term backwardation and a long-term contango is interpreted in the following way in the case of agricultural commodities: at the end of the crop year, the inventories reach their lower level. Then, the futures prices for delivery before and after the harvest reflect two different situations: there is a shortage for the deliveries before harvest, and the prices for these maturities are in backwardation. Simultaneously, prices for post-harvest delivery are in contango.

Thus, the storage theory lends itself a priori easily to an inter-temporal analysis of the prices relationship. However, when the contracts' expiration date exceeds one or two production cycles, one may ask if the explanatory factors of this theory are still of use. Indeed, some new questions arise: when the horizon of analysis increases, can the shortages on goods be something else than accidental and unpredictable? Is it possible to invoke, in the long run, some other factors than storage costs and convenience yield to explain the shape and the behavior of the prices term structure?

Gabillon (1995) is the first to give a positive answer to this last question. Its theoretical analysis reconciles the theory of normal backwardation and the storage theory. It proposes to separate the term structure of crude oil prices into two distinct segments. Each part of the curve reflects a specific economic behaviour of the operators. The first segment, corresponding to the shorter maturities (from the 1st to the 18th month), is mostly used for hedging purposes. As a result, production, consumption, stocks level and the fear of inventories disruptions are the most important explanatory factors of the prices relationship. For longer maturities however, the explanatory factors change: interest rates, anticipated inflation and the prices for competing energies determine the futures prices. In that case, the information provided by the prices is used for investment purposes. In this analysis, agents have preferred habitats: they are specialized in the holding of certain subsets of maturities and they are reluctant to alter their portfolio to take advantage of arbitrage opportunities. The latter are, therefore, leaved unexploited. Later, Lautier (2003) showed that there is a segmentation of the crude oil price curve, explained by liquidity factors, and that the segmentation evolves as the futures market matures. The segmentation is in 2003 situated around the 28th month, and not the 18th month.

Thus, the extension of the analysis to long-term horizon is relatively natural in the case of normal backwardation theory. It becomes possible providing that one quit the Keynesian framework, which is too rigid. This enlargement is not so easy in the case of the storage theory, because then, it is not sufficient to raise some simplifying assumptions. Some new explicative factors of the prices relationship must be introduced.

1.3. Dynamic analysis of the term structure

The most important feature of commodity prices curve's dynamic is probably the difference between the price behaviour of first nearby contracts and deferred contracts. The movements in the prices of the prompt contracts are large and erratic, while the prices of long-term contracts are relatively still. This results in a decreasing pattern of volatilities along the prices curve. Indeed, the variance of futures prices, and the correlation between the nearest futures price and subsequent prices decline with the maturity. This phenomenon is usually called "the Samuelson effect". Intuitively, it happens because a shock affecting the nearby contract price has an impact on succeeding prices that decreases as maturity increases (Samuelson, 1965). Indeed, as futures contracts reach their expiration date, they react much stronger to information shocks, due to the ultimate convergence of futures prices to spot prices upon maturity. These price disturbances influencing mostly the short-term part of the curve are due to the physical market, and to demand and supply shocks. Anderson (1985), Milonas (1986) and Fama and French (1987) have provided empirical support for this hypothesis for a large number of commodities and financial assets. Deaton and Laroque (1992, 1996), and Chambers and Bailey (1996) showed that the Samuelson effect is a function of the storage cost. More precisely, a high cost of storage leads to relatively little transmission of shocks via inventory across periods. As a result, futures price's volatility declines rapidly with the maturity. Lastly, in 1988, Fama and French showed that violations of the Samuelson effect might occur at shorter horizon when inventory is high. In particular, price volatilities can initially increase with the maturity of the contract, because with enough inventories, stocks-outs may not be possible for the nearest delivery months.

Figure 1 provides an illustration of the Samuelson effect. It represents the deformations of the crude oil prices term structure at different observation dates situated between March 1999 and January 2000. The futures prices correspond to the West Texas Intermediate (WTI) contract traded on the New York Mercantile Exchange (Nymex). The nearest futures prices appear as much more volatile than deferred prices. This phenomenon is especially clear between March and April 1999, when contango disappears and the curve enters in a backwardation phase. Clearly, this transformation affects mostly the short-term prices.

Compared with other commodity markets, the crude oil market displays another characteristic: it is most of the time in backwardation. This phenomenon is well known and has been widely reported (Litzenberg and Rabinowitz (1995), Edwards and Canters (1995))… This characteristic implies that the crude oil market has been extensively exploited to test the theory of normal backwardation. However, as previously noticed, the literature has failed to explain this prevalence of inverted markets in petroleum products.

Figure 1. Fluctuation of the WTI futures prices curves

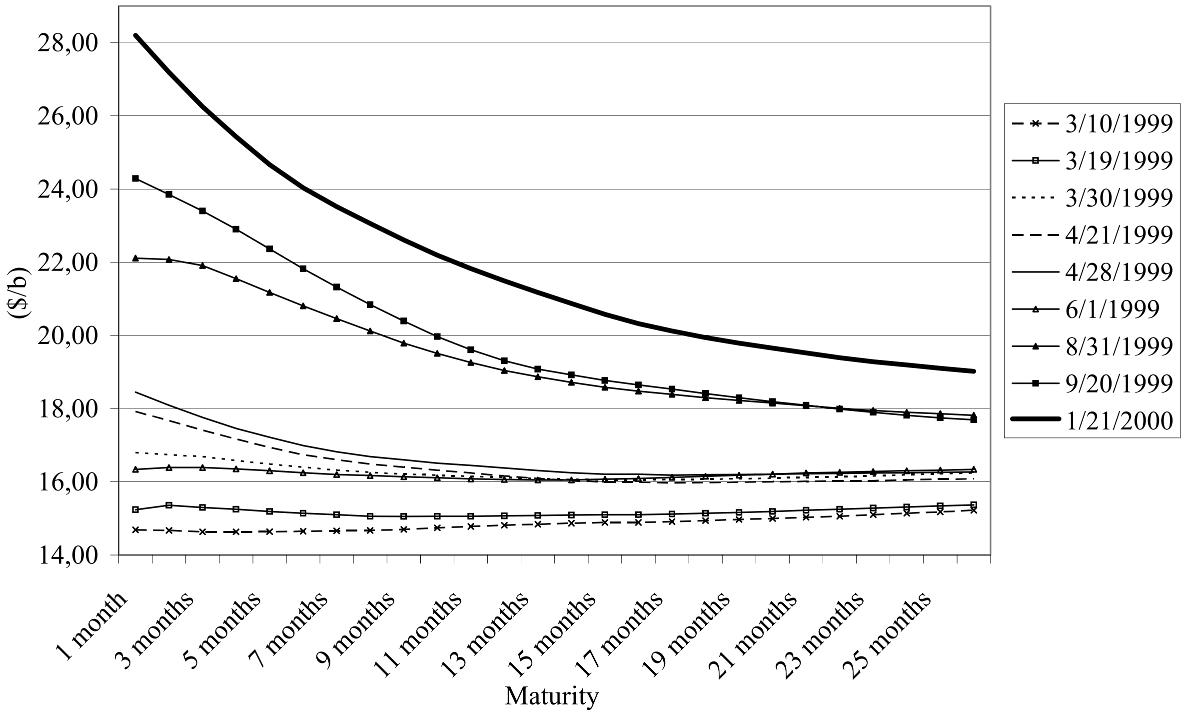
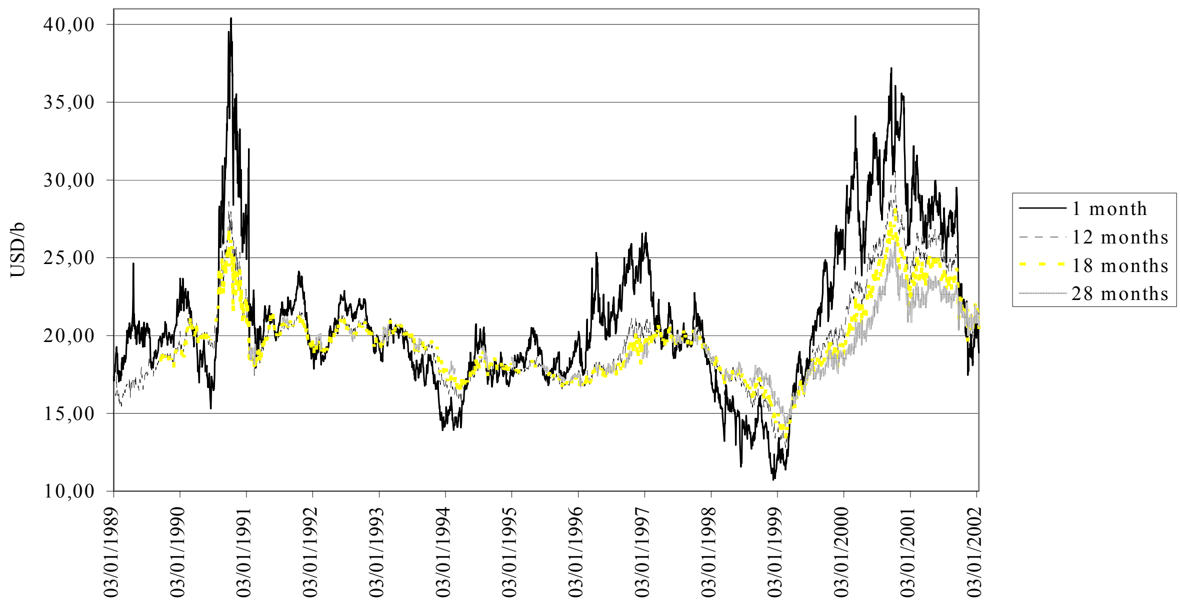


Figure 2 illustrates this characteristic of the crude oil futures market. It represents daily futures prices during 1989-2002 for selected maturities ranging from the first to the 28th months. The graphic shows that there is an alternation of backwardation and contango, and that backwardation is the most frequent situation, with some particularly strong peaks in 1990, in 1997, and in 2000.

Figure 2. Daily futures prices, 1989-2002



Another way to cope with the dynamic of futures prices is to use principal component analysis (Cortazar and Schwartz (1994), Tolmasky and Hindanov (2002), Lautier and Galli (2002)). This statistical method reduces the dimensionality of a data set by collapsing the information it contains. In data sets including many variables, groups of variables often move together because they are influenced by the same driving forces governing the behaviour of the system. In many systems, there are only a few such driving forces.

Applying a principal component analysis to crude oil futures prices' curves gives rise to three conclusions. Firstly, it leads to the identification of the type of prices curves movements, which are quite simple to describe. Three different kinds of movements can indeed be distinguished: a parallel shift in the curve (level factor), a relative shift of the curve (steepness factor) and the curvature factor. Secondly, the principal component analysis makes it possible to calculate the contribution of each component to volatility. This calculus shows that, in the case of crude oil and of copper, the first two factors account for 99% of the total variance of the futures prices. Therefore, one can consider that most of the risk associated with futures prices moves is accounted for two factors, instead of all futures prices.

SECTION 2. TERM STRUCTURE MODELS OF COMMODITY PRICES

Term structure models of commodity prices aim to reproduce as accurately as possible the futures prices observed in the market. They also provide a mean for the discovery of futures prices for horizons exceeding exchange-traded maturities. In this section, we review the main term structure models of commodity prices, from the simplest (one-factor models) to the more sophisticated versions (three factor models). Four different factors are generally used: the spot price, the convenience yield, the interest rate, and the long-term price. Before this presentation, because the models borrow to the contingent claim analysis developed for options and interest rates models, we recall the basic principles of contingent claim analysis.

2.1. Basic principles of contingent claim analysis

The standard modelling procedure for pricing commodity derivatives typically follows the contingent claim analysis developed for interest rates. The term structure models of commodity prices share three assumptions: first, the market for assets is free of frictions, taxes, or transaction costs. Second, trading takes place continuously. Third, lending and borrowing rates are equal and they are no short sale constraints.

Then, the same method as the one developed in the context of interest rates is used to construct the model. First, the state variables (namely the uncertainty sources affecting the futures price) are selected and their dynamic is specified. Then, knowing that the price of a futures contract is a function of the state variables, the time, and the contract's expiration date, using Itô's lemma makes it possible to obtain the dynamic behaviour of the futures price. Afterwards, arbitrage reasoning and the elaboration of a hedge portfolio leads to the term premium and to the fundamental valuation equation characterizing the model. Finally, whenever it is possible, the solution of the model is obtained.

The transposition of the theoretical framework developed for interest rates in the case of commodities is however not straightforward. The reasoning is indeed based on the assumption that the market is complete: in such a market a derivative asset can be duplicated by a combination of others existing assets. If the latter are sufficiently traded to be arbitrage free evaluated, they can constitute a hedge portfolio whose behaviour replicates the derivative behaviour. Their proportions are fixed such as there are no arbitrage opportunities and the strategy is risk-free. Then, in equilibrium, the return of the portfolio must be the risk free rate. The valuation is made in a risk neutral world: it does not depend on the attitude toward risk of the operators. The transposition problem arouses from the fact that commodity markets are not complete. Real markets are far from being free of arbitrage opportunities, as is the case for most financial markets. Thus, valuation will probably not be realised in a risk neutral world for commodities markets, and several risk neutral probabilities may coexist.

2.2. One-factor models

A futures price is often defined as the expectation, conditionally to the available information at a date t, of the future spot price. Indeed, the spot price is the main determinant of the futures price. Thus, most one-factor models rely on the spot price[3].

There have been several one-factor models in the literature on commodity prices. These models can be separated in step with the dynamic behaviour that is retained for the spot price. Brennan and Schwartz (1985), Gibson and Schwartz (1989 and 1990), Brennan (1991), Gabillon (1992 and 1995) use a geometric Brownian motion, whereas Schwartz (1997), Cortazar and Schwartz (1997), Routledge, Seppi and Spatt (2000) refer themselves to a mean reverting process. Moreover, the models can be distinguished in step with the assumption they retain concerning the convenience yield.

Spot price with a geometric Brownian motion

Among the different one-factor models with a geometric Brownian motion, Brennan and Schwartz's model (1985) is the most famous. It has been extensively used in subsequent research on commodity prices (see for example Schwartz (1998), Schwartz and Smith (2000), Nowman and Wang (2001), Cortazar, Schwartz and Casassus (2001), Veld-Merkoulova and de Roon (2003)).

The geometric Brownian motion is a dynamic commonly used to represent the behaviour of stock prices. When applied to commodities, the spot price's dynamic is the following:

$$dS(t) = \mu S(t)dt + \sigma_S S(t)dz \qquad [1]$$

where:

- S is the spot price
- μ is the drift of the spot price
- σ_S is the spot price volatility
- dz is an increment to a standard Brownian motion associated with S.

The use of this representation implies that the variation of the spot price at t is supposed to be independent of the previous variations, and that the drift μ conducts the price's evolution. The uncertainty affecting this evolution is proportional to the level of the spot price: when stocks are rare, S is high; in this situation, any change in the demand has an important impact on the spot price, because inventories are not sufficiently abundant to absorb the prices' fluctuations.

An arbitrage reasoning and the construction of a hedging portfolio leads to the fundamental valuation equation of the futures prices, which is:

[3] One exception is the one-factor model relying on the convenience yield developed by Veld-Merkoulova and de Roon (2003).

$$\frac{1}{2}\sigma_S^2 S^2 F_{SS} + (r-c)SF_S - F_\tau = 0 \qquad [2]$$

where c is the convenience yield and r is the interest rate. These two parameters are supposed to be constant. The convenience plays the role of a stochastic dividend in the spot price process.

The terminal boundary condition associated with this equation is:

$$F(S,T,T) = S(T)$$

It represents the convergence of the futures and the spot prices at the contract's expiration. This convergence is due to the possibility to deliver the commodity at maturity. It is insured by arbitrage operations between the physical and the paper markets.

The solution of the model expresses the relationship at t between an observable futures price F for delivery in T and the state variable S. It is the following[4]:

$$F(S,t,T) = Se^{(r-c)\tau} \qquad [3]$$

where τ is the maturity of the contract: $\tau = T - t$.

Brennan and Schwartz's article also fixes the definition of the convenience yield. Since then indeed, authors always refer to this definition:

> *"The convenience yield is the flow of services that accrues to an owner of the physical commodity but not to the owner of a contract for future delivery of the commodity. [...] Recognizing the time lost and the costs incurred in transporting a commodity from one location to another, the convenience yield may be thought of as the value of being able to profit from temporary local shortages of the commodity through ownership of the physical commodity. The profit may arise either from local price variations or from the ability to maintain a production process as a result of ownership of an inventory of raw material"*[5].

Brennan and Schwartz's model is probably the most simple term structure model of commodity prices. Consequently, it is very popular. However, the geometric Brownian motion is probably not the best way to represent the price dynamic. Indeed, the storage theory and the Samuelson effect show that the mean reverting process is probably more relevant.

Mean reverting process

Among the different one-factor models retaining the mean reverting process, Schwartz' one (1997), inspired by Ross (1995), is probably the most famous[6]. In that case, the dynamic of the spot price is the following:

$$dS = S\kappa(\mu - \ln S)dt + \sigma_S S dz_S$$

where:
- S is the spot price
- κ is the speed of adjustment of the spot price,
- μ is the long run mean log price,
- σ_S is the spot price volatility
- dz_S is an increment to a standard Brownian motion associated with the spot price.

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4 The solution of the models can be obtained with a Feynman-Kac solution.

5 Brennan and Schwartz, 1985.

6 This model was also used by Schwartz and Smith, 2000.

In this situation, the spot price fluctuates around its long run mean. The presence of a speed of adjustment insures that the state variable will always return to its long run mean μ. Therefore, two factors influence the spot price behaviour. First, it has a propensity to return to its long run mean. Second, it is simultaneously volatile and random shocks can move it away from μ.

The use of a mean reversion process for the spot price makes it possible to take into account the behaviour of the operators in the physical market. When the spot price is lower than its long run mean, the industrials, expecting a rise in the spot price, reconstitute their stocks, whereas the producers reduce their production rate. The increasing demand on the spot market and the simultaneous reduction of supply have a rising influence on the spot price. Conversely, when the spot price is higher than its long run mean, industrials try to reduce their surplus stocks and producers increase their production rate, pushing thus the spot price to lower levels.

This formulation of the spot price behaviour is preferable to the geometric Brownian motion, but it is not perfect. For example, the mean reverting process does not exclude that the state variable become negative. The same critic was addressed to this stochastic process in the case of interest rates. Moreover, the storage theory shows that in commodity markets, the basis does not behave similarly in backwardation and in contango. The mean reverting process previously presented does not allow taking into account that characteristic.

The mean reverting process was also used by Cortazar and Schwartz in 1997, in a more sophisticated model. Indeed, the authors introduce a variable convenience yield that depends on the deviation of the spot price to a long-term average price.

Other one-factor models

Among the other one-factor models, the models studied by Brennan in 1991 are quite interesting because all of them rely on a specific hypothesis concerning the convenience yield, which is an endogenous variable. The first model is the one developed with Schwartz in 1985. In that case, the convenience yield is a simple linear function of the spot price. The second model expresses the convenience yield as a non-linear function of the price:

$$C(S) = a + bS + cS^2$$

This formula is chosen because it is more flexible than the one used in 1985.

A third version underlines that when the convenience yield is low, it cannot be lower than the opposite of the storage cost. The latter is supposed to be constant for a large spread of stocks levels. Finally, in this model, the level of the convenience increases with the spot price:

$$C(S) = \max (a, b + cS)$$

Brennan's empirical study, as well as Gibson and Schwartz's article (1990), lead however to the conclusion that there are limits to one-factor models.

2.3. Two-factor models

The homogeneity in the choice of the state variables disappears when a second stochastic variable is introduced in term structure models of commodity prices. Most of the time, the second state variable is the convenience yield. However, models based on long-term price or on volatility of the spot price have also been developed. In all these models, the introduction of a second state variable allows obtaining richer shapes of curves than one-factor models (especially for long term maturities) and richer volatility structures. This improvement is rather costly, however, because two-factor models are more complex.

The convenience yield as the second state variable

Schwartz's model (1997) is probably the most famous term structure model of commodity prices. It was used as a reference to develop several models that are more sophisticated (Hilliard and Reis (1998), Schwartz (1998), Neuberger (1999), Schwartz and Smith (2000), Lautier and Galli (2001), Yan (2002), Richter and Sorensen (2002), Veld-Merkoulova and de Roon (2003)). It is inspired by the one proposed by Gibson and Schwartz in 1990. Compared with its former version, the latest model is more tractable because it has an analytical solution.

The two-factor model supposes that the spot price S and the convenience yield C can explain the behavior of the futures price F. The dynamics of these state variables is:

$$\begin{cases} dS = (\mu - C)Sdt + \sigma_S S dz_S \\ dC = [k(\alpha - C)]dt + \sigma_C dz_C \end{cases} \qquad [4]$$

with: κ, σ_S, σ_C >0

where: - μ is the drift of the spot price,

- σ_S is the spot price volatility,
- dz_S is an increment to a standard Brownian motion associated with S,
- α is the long run mean of the convenience yield,
- κ is the speed of adjustment of the convenience yield,
- σ_C is the volatility of the convenience yield,
- dz_C is an increment to a standard Brownian motion associated with C.

In this model, the convenience yield is mean reverting and it intervenes in the spot price dynamic. The Ornstein-Uhlenbeck process relies on the hypothesis that there is a regeneration property of inventories, namely that there is a level of stocks, which satisfies the needs of industry under normal conditions. The behaviour of the operators in the physical market guarantees the existence of this normal level. When the convenience yield is low, the stocks are abundant and the operators sustain a high storage cost compared with the benefits related to holding the raw materials. Therefore, if they are rational, they try to reduce these surplus stocks. Conversely, when the stocks are rare the operators tend to reconstitute them.

Moreover, as the storage theory showed it, the two state variables are correlated. Both the spot price and the convenience yield are indeed an inverse function of the inventories level. Nevertheless, as Gibson and Schwartz (1990) demonstrated it, the correlation between these two variables is not perfect. Therefore, the increments to standard Brownian motions are correlated, with:

$$E[dz_S \times dz_C] = \rho dt$$

where ρ is the correlation coefficient.

An arbitrage reasoning and the construction of a hedging portfolio leads to the solution of the model. It expresses the relationship at t between an observable futures price F for delivery in T and the state variables S and C. This solution is:

$$F(S,C,t,T) = S(t)\times \exp\left[-C(t)\frac{1-e^{-\kappa\tau}}{\kappa} + B(\tau)\right] \qquad [5]$$

with: - $B(\tau)=\left[\left(r-\widehat{\alpha}+\frac{\sigma_C^2}{2\kappa^2}-\frac{\sigma_S\sigma_C\rho}{\kappa}\right)\times\tau\right]+\left[\frac{\sigma_C^2}{4}\times\frac{1-e^{-2\kappa\tau}}{\kappa^3}\right]+\left[\left(\widehat{\alpha}\kappa+\sigma_S\sigma_C\rho-\frac{\sigma_C^2}{\kappa}\right)\times\left(\frac{1-e^{-\kappa\tau}}{\kappa^2}\right)\right]$

- $\widehat{\alpha}=\alpha-(\lambda/\kappa)$

where: - r is the risk free interest rate, assumed constant,

- λ is the market price of convenience yield risk,

- $\tau = T - t$ is the maturity of the futures contract.

This formulation, which is quite tractable, presents a limit. Indeed, it ignores that in commodity markets, price's volatility is positively correlated with the degree of backwardation. This phenomenon has been widely commented and reported (see for example Williams and Wright (1991), Ng and Pirrong (1994), Litzenberg and Rabinowitz (1995)…) and it can be explained by the examination of arbitrage relationships between the physical and the futures markets. Such a study shows that the basis has an asymmetrical behavior: in contango, its level is limited to storage costs. This is not the case in backwardation:

"*Arbitrage can always be relied upon to prevent the forward price from exceeding the spot price by more than net carrying cost... [but] can not be equally effective in preventing the forward price from exceeding the spot price by less than net carrying cost.*"[7]

Furthermore, the basis is stable in contango, and volatile in backwardation, since in this situation stocks cannot absorb price fluctuations. This phenomenon leads sometimes to consider that the convenience yield is an option (Heinkel, Howe and Hughes (1990), Milonas and Tomadakis (1997), Milonas and Henker (2001)) or that it has an asymmetrical behavior. This assumption has been introduced in term structure models by Brennan (1991), Routledge, Seppi, and Spatt (2000), and Lautier and Galli (2001).

Brennan (1991) introduces an asymmetric convenience yield in his model because he takes into account a non negativity constraint on inventory. However, he supposes that the convenience yield is deterministic (this one-factor model was previously presented). In the model presented by Routledge *et alii*, the asymmetry in the behavior of the convenience yield is introduced in the correlation between the spot price and the convenience yield. This correlation is higher in backwardation than in contango. In this model, the convenience yield is an endogenous variable, determined by the storage process. However, it is stochastic. The two factors are the spot price and exogenous transitory shocks affecting supply and demand. Lautier and Galli (2001) propose a two-factor model inspired by Schwartz's model (1997), where the convenience yield is also mean reverting and acts as a continuous dividend. An asymmetry is however introduced in the convenience yield dynamic: it is high and volatile in backwardation, when inventories are rare. It is conversely low and stable when inventories are abundant. The asymmetry is measured by the parameter β. When the latter is set to zero, the asymmetrical model reduces to Schwartz's model.

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[7] Blau (1944).

The long-term price as a second state variable

Another approach of the term structure of commodity prices consists in considering the decreasing pattern of volatilities along the prices curve. In that situation, it is possible to infer that the two state variables are the extremities of the prices curve, namely the spot price and the long-term price. This kind of approach was followed by Gabillon (1992) and Schwartz and Smith (2000).

Gabillon (1992) uses the spot and the long-term prices as state variables. In this model, the convenience yield is an endogenous variable, which depends on the two factors. The use of the long-term price as a second state variable is justified by the fact that the long-term price can be influenced by elements that are exogenous to the physical market, such as expected inflation, interest rates, or prices for renewable energies. Thus, the spot and the long-term prices reassemble all the factors allowing the description of the term structure movements. The author retains a geometric Brownian motion to represent the behaviour of the long-term price. Moreover, the two state-variable are assumed positively correlated.

Schwartz and Smith (2000) propose a two-factor model that allows mean reversion in short-term prices and uncertainty in the equilibrium level to which prices revert. Those factors are not directly observables, but they are estimated from spot and futures prices. Movements in prices for long-maturity futures contracts provide information about the equilibrium price level, and differences between the prices for short and long-term contracts provide information about short-term variations in prices. This model does not explicitly consider changes in convenience yields over time, but it is equivalent to the two-factor model proposed by Gibson and Schwartz in 1990, in that the state variables in one of the models can be expressed as linear combinations of the state variables in the other model.

The authors decompose the spot prices into two stochastic factors:

$$\ln(S_t) = \chi_t + \xi_t$$

where : - S_t is the spot price at *t,*

- χ_t is the short-term deviation in prices,
- ξ_t is the equilibrium price level.

The short-term deviation is assumed to revert to zero, following an Ornstein-Uhlenbeck process, and the equilibrium level is assumed to follow a Brownian motion process. The dynamic of these two state variables is the following:

$$\begin{cases} d\chi_t = -\kappa\chi_t dt + \sigma_\chi dz_\chi \\ d\xi_t = \mu dt + \sigma_\xi dz_\xi \end{cases} \qquad [6]$$

where: - κ is the speed of adjustment of the short-term deviation,

- σ_χ is the volatility of the short-term prices,
- dz_χ is an increment to a standard Brownian motion associated with χ_t,
- μ is the drift of the equilibrium price level,
- σ_ξ is the volatility of the equilibrium price level,
- dz_ξ is an increment to a standard Brownian motion associated with ξ_t

Changes in the short-term deviation represent temporary changes in prices (*e.g.* caused by abrupt weather alteration, supply disruption, etc) and are not expected to persist. They are tempered by the ability of market participants to adjust inventory levels in response to changing market conditions.

Changes in the long-term level represent fundamental modifications, which are expected to persist. The latter are due to changes in the number of producers in the industry, and the long-term equilibrium is also determined by expectations of exhausting supply, improving technology for the production and discovery of the commodity, inflation, as well as political and regulatory effects.

The most important advantage of this model is that it avoids the questions concerning the convenience yield, its estimation, and its economic significance (on this particular point, see for example Williams and Wright (1989), Brennan, Williams and Wright (1997), Frechette and Fackler (1999)). The idea of a long-term equilibrium is also in line with recent works on long memory in the commodity futures markets. Long memory in convenience yields has been studied by Mazaheri (1999), and Barkoulas, Labys and Onochie (1999) showed that there is long memory in futures prices. Long memory or long-term dependence describes the correlation structure of series at long lags. If a series exhibits long memory, there is persistent temporal dependence between distant observations. Such series are characterized by distinct but non-periodic cyclical patterns. However, this new model also introduces a new problem: is it interesting to represent equilibrium with a stochastic variable?

Seasonality

Apart from the nature of the state variables, some research has been conducted on the seasonality of commodity prices. In this field, Gabillon once again opened the road in 1992 with a model including a seasonal function as a composite of sine and cosine functions. The same type of formalization was retained by Richter and Sorensen (2002). However, the latter model takes the spot price and the convenience yield as state variables, whereas Gabillon retains the spot and the long-term prices. Moreover, Richter and Sorensen worked on futures prices for soybeans, whereas Gabillon was concerned by petroleum products.

2.4. Three-factor models

Until 1997, every term structure model of commodity prices assumed the interest rate constant. Such a hypothesis amounts saying that the term structure of interest rates is flat. This is all the more reductive as the horizon of analysis is remote. Schwartz, in 1997, proposes a model including three state variables: the spot price, the convenience yield and the interest rate. The latter has a mean reverting behaviour. More precisely, the dynamic of the state variables is the following:

$$\begin{cases} dS = S(r-C)dt + \sigma_S dz_S \\ dC = \kappa(\alpha - C)dt + \sigma_C dz_C \\ dr = a(m-r)dt + \sigma_r dz_r \end{cases} \qquad [7]$$

where: - μ is the drift of the spot price,

- α is the long run mean of the convenience yield,
- κ is the speed of adjustment of the convenience yield,
- a is the speed of adjustment of the interest rate,
- m is the long run mean of the interest rate,
- σ_i is the volatility of the variable *i*,
- dz_i is an increment to a standard Brownian motion associated with the variable *i*.

The introduction of a stochastic interest rate in the analysis of prices relationship is important on a theoretical point of view: the assumption of a constant interest rate amounts saying that futures and forward prices are equivalent, which is not the case[8]. With a stochastic interest rate, it is possible to take into account the margin calls mechanism of the futures market. Thus, two distinct pay-off structures can be taken into account for futures and forward contracts. Finally, the presence of the interest rate as a third explaining factor is consistent with the storage theory.

Since 1997, several three-factor models were proposed. In 1998, Hilliard and Reis introduced jumps in the spot price process, in order to take into account the large and abrupt changes, due to supply and demand shocks, that affect certain commodity markets, especially the energy commodities used for heating. They modified the three models proposed by Schwartz in 1997. In 2000, Schwartz and Smith proposed an extension of their short-term/long-term model in which the growth rate for the equilibrium price level is stochastic. This third model improves the model's ability to fit long-term futures prices. Another improvement of three factors models was proposed by Yan (2002). In his model, he incorporates stochastic convenience yields, stochastic interest rates, stochastic volatility and simultaneous jumps in the spot price and volatility. The convenience yield follows an Ornstein-Uhlenbeck process, whereas the interest rate follows a square-root process, and the volatility follows a square-root jump-diffusion process. However, the author finds that stochastic volatility and jumps do not alter the futures price at a given point in time. Nevertheless, they play important roles in pricing options on futures. Lastly, Cortazar and Schwartz (2003) proposed a three-factor model related to Schwartz (1997), where all three factors are calibrated using only commodity prices[9]. In this model, the authors consider as a third risk factor the long-term spot price return, allowing it to be stochastic and to mean revert to a long-term average. The two others stochastic processes are the spot price and the convenience yield. The convenience yield models temporary variations in prices due to changes in inventories, whereas the long-term returns models long-term variations due to changes in technologies, inflations or demand pattern.

In practice, the development of three factor models arises the question of the arbitrage between reality and simplicity. Although the introduction of a third factor may improve the performances of the models in terms of their ability to describe the stochastic evolution of futures prices, there is always a balance to find between the fidelity of the prices models and the need for parsimony, especially when the models are conceived for the evaluation of more complex derivatives products.

SECTION 3. TERM STRUCTURE MODELS AND THE DESCRIPTION OF PRICES CURVES

This section reviews the main empirical results obtained with term structure models of commodity prices. First of all, simulations highlight the influence of the assumptions concerning the stochastic process retained for the state variables and the number of state variables. Then, in order to test the model, parameters values are needed. Thus, the method generally used for the estimation of the parameters is exposed. Lastly, the models' performances, namely their ability to reproduce the term structure of commodity prices, are presented.

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[8] Cox, Ingersoll, Ross, (1981).

[9] In 1997, Schwartz calibrated the third factor of his model using bond prices instead of commodity futures prices.

3.1. Simulations

Simulations make it possible to compare the prices curves extracted from the models and to appreciate how realistic the models are. Among the different term structure models presented below, two are retained for simulation purposes: Brennan and Schwartz's model (1985) and two-factor Schwartz's model (1997). These models are probably the most well known nowadays and they are extensively used. The values retained for the simulations are inspired by empirical tests carried out on the crude oil market[10]. They are presented in Table 1.

Table 1. Values retained for the parameters

	S	C	r	α	κ	σ_S	σ_C	ρ	λ
Min	12	-0.3	0.02	-0.1	0.5	0.1	0.3	0	-1
Max	20	0.2	0.06	0.1	2	0.5	0.7	1	1

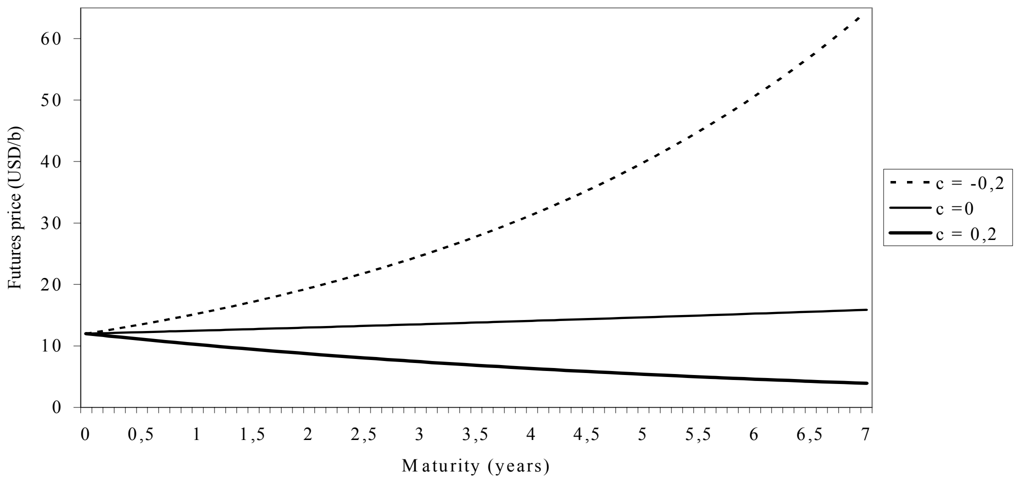
Brennan and Schwartz's model (1985)

In Brennan and Schwartz's model, the relative value of the two parameters (the interest rate r and the convenience yield c) determine the whole shape of the term structure of futures prices. When the interest rate is superior to the convenience yield, the curve is in contango. Conversely, the curve is in backwardation. Figure 3 illustrates that with this model, prices curves can be only monotonically decreasing, monotonically increasing, or flat. The growth rate of the futures price is indeed a constant:

$$\delta F / (F \delta \tau) = r - c$$

When the difference *(r-c)* is positive (as is the case when c is set to -0.2) prices are upward sloping and they can reach a level without real economic significance: almost USD 65 per barrel for a seven years expiration date. When conversely the growth rate is negative, the curve is in backwardation and prices tend towards zero.

Figure 3. Brennan and Schwartz's model, impact of a variation in the convenience yield
S = 12 ; r = 0.04



Moreover, this model considers that the convenience yield is a constant, and it supposes that the volatility of the returns is the same for all the maturities:

$$\delta F / F = \sigma_S dz$$

[10] Schwartz (1997), Lautier and Galli (2001), Lautier and Galli (2002).

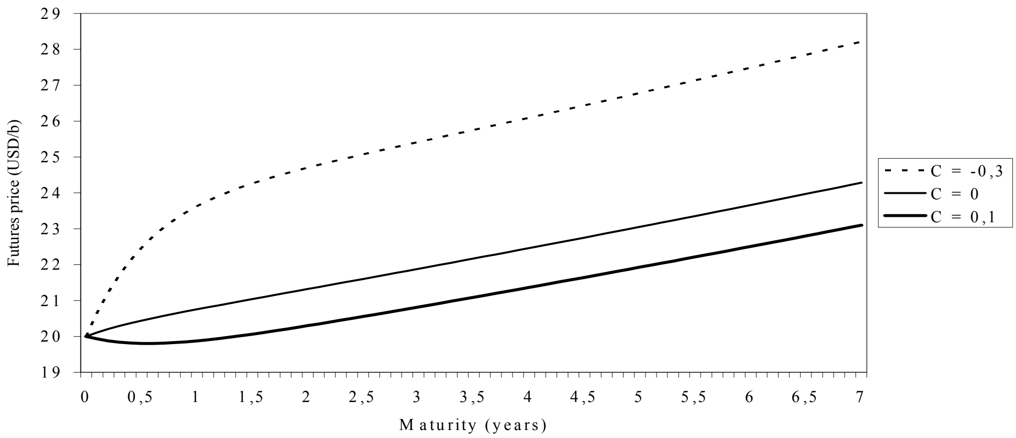
Therefore, Brennan and Schwartz's model presents important drawbacks, especially for long-term analysis. However, its simplicity renders it very tractable and it is always used today.

Schwartz's model (1997)

With Schwartz's model, we can obtain various shapes of prices curves, as is shown in Figure 4, that presents simulations with different values of convenience yields. Indeed, the curves can be sunken ($C = 0.1$), humped ($C = -0.3$), or flat ($C = 0$). The level of the futures prices is a decreasing function of the convenience yield: the more it increases, the more the price level diminishes.

Figure 4. Schwartz's model, impact of a variation in the convenience yield

$S = 20$; $r = 0.06$; $\alpha = 0$; $\kappa = 2$; $\rho = 0.9$; $\sigma_S = 0.3$; $\sigma_C = 0.4$; $\lambda = 0$

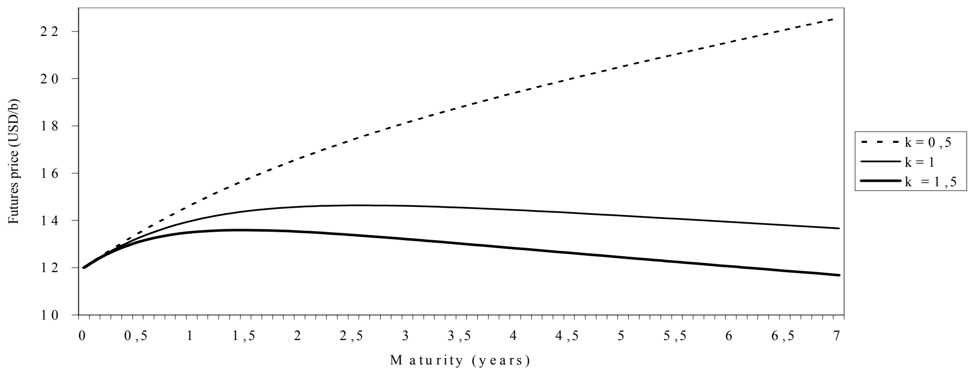


The simulations also show that the gap between the convenience yield and its long run mean α influences the shape of the prices curves. When the convenience yield is far from its long run mean ($C = -0.3$ and $\alpha = 0$), it takes 3.4 years until the curve becomes stable. When conversely the convenience yield is equal to its long run mean ($C = \alpha = 0$), the growth rate of the futures prices becomes stable as early as 1.8 year.

Another interesting series of simulations can be made with the speed of adjustment of the convenience yield. The latter are illustrated by figure 5. They show that the growth rate of the futures prices rises when the speed of adjustment diminishes. The gap between the nearest and the remote maturities increases when mean reversion reduces.

Figure 5. Schwartz's model, impact of a variation in the speed of adjustment

$S = 12$; $C = -0.2$; $r = 0.06$; $\alpha = 0.1$; $\rho = 0$; $\sigma_S = 0.1$; $\sigma_C = 0.2$; $\lambda = 0$



Thus, the introduction of a second state variable makes it possible to obtain richer prices curves than with a one-factor model. Beyond this remark, which is also valuable in the field of interest rates, Schwartz's model is also more realistic that Brennan and Schwartz's model because in the two-factor model, the volatility of the futures prices decreases with the maturity τ:

$$\sigma_F^2(\tau) = \sigma_S^2 + \sigma_C^2\left(\frac{1-e^{-\kappa\tau}}{\kappa}\right)^2 - \left[2\times\frac{1-e^{-\kappa\tau}}{\kappa}\times\rho\sigma_S\sigma_C\right]$$

When the contract reaches its expiration date, the futures price's volatility converges towards the spot price's volatility. Conversely, the volatility of the futures prices tends towards a fixed value when the maturity tends towards infinity:

$$\lim_{\tau\to\infty}\sigma_F^2 = \sigma_S^2 + \frac{\sigma_C^2}{\kappa^2} - \frac{2\rho\sigma_S\sigma_C}{\kappa}$$

However, the two-factor model it is also more complex, because it includes six parameters, whereas the one-factor model has only two of them.

3.2. Parameters' estimation

To assess the performances of a model, parameters values are needed. They are necessary to compute the estimated futures prices and to compare them with empirical data. The parameters estimation is not obvious though, because many term structure models rely on non-observable state variables. In order to cope with this difficulty, a method that was proposed by Schwartz in 1997, namely a Kalman filter, can be used.

Non-observable state variables and their approximations

The non-observable state variables are, in the case of term structure models of commodity prices, the spot price, the convenience yield, and the long-term equilibrium price level.

The non-observable nature of the spot price signifies that this problem concerns all the models. The spot price is regarded as non observable because in most commodities markets there is a lack of reliable time series of spot prices: physical markets are geographically dispersed, transactions are not standardized, the prices reporting mechanism does not enforce the operators to disclose their transactions prices, etc. In the case of the American crude oil market, spot prices are also affected by another problem specific to the petroleum used as the underlying commodity of the futures contract. Firstly, the transactions volume for the West Texas Intermediate's quality is very low and the spot prices provide information only on local supply and demand. Secondly, there are difficulties associated with the delivery procedure of the physical product. The WTI is delivered in the Mexican Gulf and there are sometimes problems due to an under-capacity of the pipeline system, which can create prices jumps. The latter are due to the delivery system and no to general market conditions. This phenomenon, reported by Horsnell and Mabro (1993), is known as the "Cushing Cushion", because usually problems arise at Cushing. The usual way to cope with the non-observable nature of the spot price consists in retaining the nearest futures price as an approximation.

The convenience yield is also a non-observable variable because it does not correspond to a traded asset. The approximation method usually chosen for this variable consists in using the solution of Brennan and Schwartz's model (1985). The calculus requires the use of two prices: the nearest and the subsequent futures prices. Let us denote the maturities of these prices as T_1 and T_2. The convenience yield c is then:

$$c = r - \frac{\ln\left(F(S,t,T_1)\right) - \ln\left(F(S,t,T_2)\right)}{T_1 - T_2}$$

Lastly, the long-term price present the same characteristic than the convenience yield: it is not a traded asset. To cope with this difficulty, Schwartz and Smith (2000) use Kalman filtering techniques, which were proposed by Schwartz in 1997.

Kalman filters

The main principle of the Kalman filters is to use temporal series of observable variables in order to reconstitute the values of non-observable variables. In finance, the problem of non-observable variables is not characteristic of commodity prices. It also arises with term structure models of interest rates, with market portfolios in the capital asset pricing model, with credit risk, etc. When associated with an optimization procedure, the Kalman filter provides a way to estimate the model parameters. Finally and most importantly, because it is very fast, the method is also interesting for large data sets.

There are different versions of Kalman filters[11]. The simple one is also the most famous and it is quite frequently used in finance nowadays[12]. Nevertheless, it is not suitable for nonlinear models. In that case, an extended filter can be used (Javaheri *et al*, (2003)). However, the latter relies on an approximation that influences the model performances: the extended filter leads to less precise results than the simple one. Nevertheless, it is still acceptable in the case of term structure models of commodity prices. Apart from the linearization, the two filters rely on the same principles.

The Kalman filter is an iterative process. To use it, the model has to be expressed in a state-space form characterized by a transition equation and a measurement equation[13]. The transition equation describes the dynamics of the state variables $\tilde{\alpha}$, for which there are no empirical data. During the first step of the iteration – the prediction phase – this equation is used to compute the values of the non-observable variables at time t, conditionally on the information available at time *(t-1)*. The predicted values $\tilde{\alpha}_{t/t-1}$ are then substituted into the measurement equation to determine the value of the measures $\tilde{y}_t$. The measurement equation represents the relationship linking the observable variables $\tilde{y}$ with the non-observable $\tilde{\alpha}$. In the second iteration step – or innovation phase – the innovation v_t, which is the difference, at t, between the measure $\tilde{y}_t$ and the empirical data y_t, is calculated. The innovation is used, in the third iteration step – or updating phase – to obtain the value of $\tilde{\alpha}_t$ conditionally on the information available at t. Once this calculation has been made, $\tilde{\alpha}_t$ is used to begin a new iteration. Thus, the Kalman filter makes it possible to evaluate the non-observable variables $\tilde{\alpha}$, and it updates their value in each step using the new information.

This brief presentation explains why the Kalman filter is a very fast method. Indeed, to reconstitute the temporal series of the non-observable variables, only two elements are necessary: the transition equation and the innovation v. Because there is an updating phase in the iteration, very little information is needed.

[11] For a presentation of Kalman filters, see for example Harvey (1989).

[12] See for example Schwartz (1997) or Babbs and Nowman (1999).

[13] There is more than one state-space form for some models. Because some of them are more stable, the choice of a specific representation is important.

3.3. Models' performances

The performances of a model measure its ability to reproduce the term structure of commodity prices. To assess these performances, criteria values are needed. We first present these criteria. Then, we expose the main empirical results obtained with the models.

Performances criteria

Two criteria are usually retained to measure the performances of a term structure model: the mean pricing error (MPE) and the root mean squared error (RMSE):

The MPE is defined as follows:

$$MPE = \frac{1}{N}\sum_{n=1}^{N}\left(\widetilde{F}(n,\tau) - F(n,\tau)\right)$$

where N is the number of observations, $\widetilde{F}(n,\tau)$ is the estimated futures price for maturity τ at the date n, and $F(n,\tau)$ is the observed futures price. The MPE measures the estimation bias for a given maturity. If the estimation is good, the MPE should be very close to zero.

Using the same notation, the RMSE is, for a given maturity τ:

$$RMSE = \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(\widetilde{F}(n,\tau) - F(n,\tau)\right)^2}$$

The RMSE can be considered as an empirical variance, which measures the estimation stability. This second criterion is considered as more representative because price errors can offset themselves and the MPE can be low even if there are strong deviations.

Empirical results

The empirical tests carried out with term structure models share some general features, which are firstly presented. Then, we expose the empirical results obtained with one-factor, two-factor and three-factor models.

General features

The term structure models share two general features. Firstly, parameters change with the study period (Schwartz 1997). Thus, when applying the term structure models, parameters should be recomputed regularly. This can become a problem if the model has no analytical solution, because of the computing time. Secondly, parameters vary with the maturity (Schwartz 1997, Lautier 2003). When the convenience yield is stochastic and characterized by a mean reverting process, its speed of adjustment is a decreasing function of the maturity. Indeed, mean reversion concerns the inventories, which are of little importance for long-term maturities. The same kind of explanation can be evoked for the spot price volatility and the convenience yield volatility: their level decreases with the maturity because when this happens, the shocks on supply and demand have a lowest impact on the futures prices. These parameters changes lead to the conclusion that ideally, the parameters of the term structure models should depend on the maturity.

One-factor models

The different empirical tests carried out with one-factor models generally lead to the conclusion that the models perform badly. This result, however, does not apply systematically for all the commodities. In 1991, comparing the empirical results obtained with three one-factor models for several commodity markets, Brennan showed that the convenience yield is close to zero for the precious metals and is positive for industrial commodities. He interprets that phenomenon as the result of differences in the motivations of the operators holding precious metals and industrial commodities. Precious metals are essentially possessed for speculative reasons. Inventories are high and they constitute a reserve of value rather than the input of a production process. In that situation, the storage cost is very low compared with the stocks value. Therefore, the convenience yield plays a marginal role for these specific categories of commodities, and one-factor models are suited for them.

Schwartz, in 1997, confirms this empirical finding. He also validates the assumption of a mean reverting process for the spot price: indeed, he shows that, on the crude oil and on the copper markets, the parameter representing the speed of adjustment is statistically significant. Moreover, this dynamic is suited for industrial commodities, but it is not a good way to represent the spot price behaviour in the case of gold.

Two-factor models

Several empirical tests testify the superiority of two-factors models on one-factor models: Brennan (1991), Schwartz (1997), Schwartz and Smith (2000). In all these cases, the models performances are strongly improved by the introduction of a second state variable. The tests also show that the mean reverting process is suited to represent the dynamic behaviour of the convenience yield (Brennan (1991), Schwartz (1997)). The parameter representing the speed of adjustment is significantly different from zero. The same result is obtained for the long run mean of the convenience yield.

The ability to reproduce the shape of the term structure of commodity prices is satisfying with a two-factor model. An example of this fidelity to reality is given by Table 2, which represents the performances of Schwartz's model on the crude oil market, from 1998 to 2001, for futures prices of different maturities: one month, three months, six months and nine months.

Table 2. Performances of Schwartz's model on the crude oil market, 1998-2001[14].

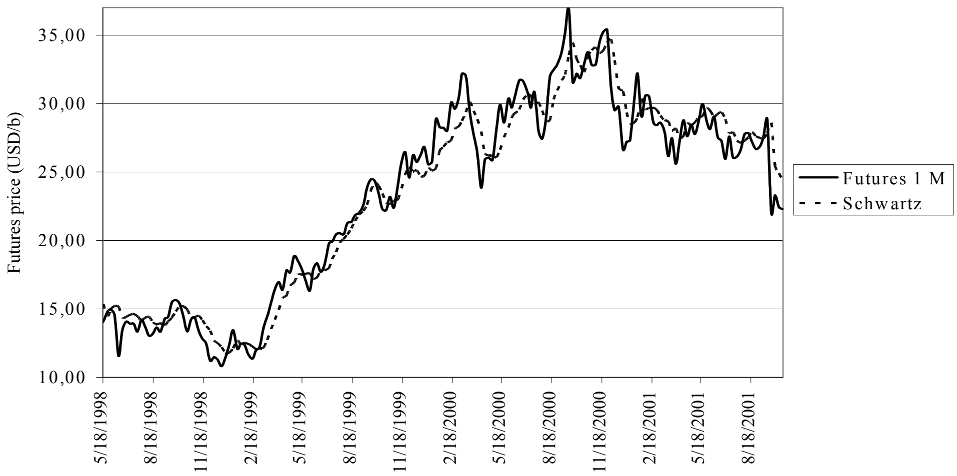
Maturity	MPE	RMSE
1 month	-0.0604	2.3197
3 months	-0.1078	1.9894
6 months	-0.0545	1.7152
9 months	-0.0073	1.5675
Average	-0.0575	1.8980

Thus, the performances can be very good: the average MPE is around 6 cents on the period. Moreover, they are still excellent when the maturity of the contracts is extended until seven years (Lautier, 2003). A graphical representation also shows that the model is able to reproduce the prices dynamic quite precisely even if, like in 1998-2001, there are very large fluctuations in the futures prices. Figure 6

[14] Extracted from Lautier and Galli (2002).

shows the results obtained for the one month's futures prices. During that period, the crude oil price jumps from USD 11 per barrel to USD 37 per barrel.

Figure 6. Estimated and observed futures prices for the one month maturity, 1998-2001



Finally, it is possible to underline the model's ability to reproduce the evolution of prices curves through time. Figure 7 represents six term structures of crude oil prices, for different maturities, observed weekly on the Nymex between the 9th of August and the 14th of September 1999. During this period, the prices curves are always in backwardation, and they are characterized by the presence of a little bump. Moreover, the intensity of the backwardation increases and the curve goes higher, as the futures prices for all maturities rise.

Figure 7. Observed term structures of crude oil prices

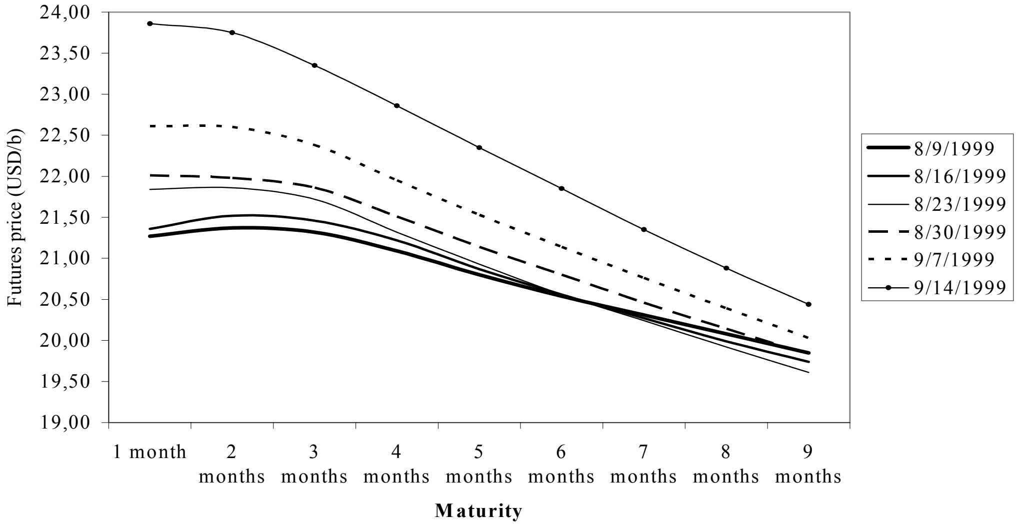
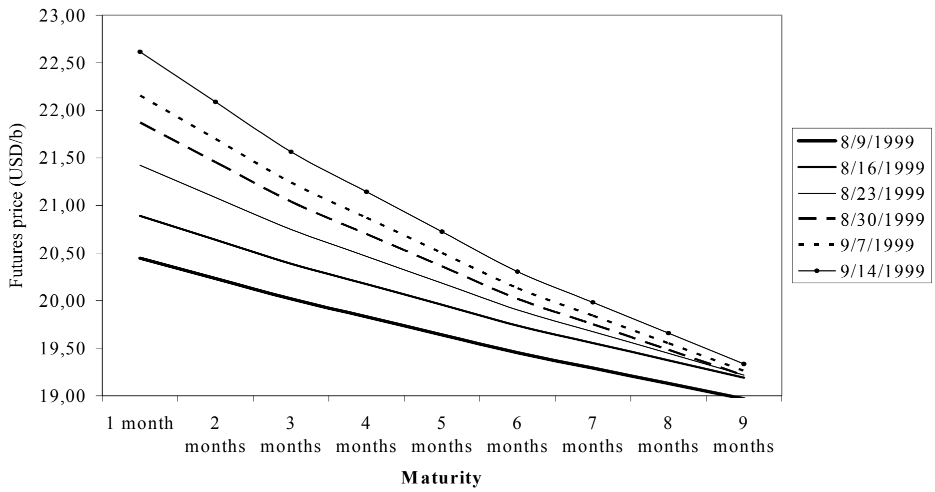


Figure 8 shows how the model reproduces this evolution. It represents, for the same observations dates, the estimated term structure of crude oil prices. The model is able to replicate correctly not only the displacement towards the heights, but also the slope's intensification. However, despite it is theoretically able to do it, the model doesn't represent, in this example, the little bump in the curves that is observed empirically.

Figure 8. Estimated term structure of crude oil prices



Empirical tests (Lautier and Galli, 2001) also showed that the introduction of an asymmetry in the convenience yield behaviour improves the performances of the model.

All models including the convenience yield as a second state variable testify a high correlation between the spot price and the convenience yield. Most of the time, the level of the correlation coefficient attains 0.9 for Schwartz's model[15]. Therefore, the correlation is so high that one may ask if the choice of the convenience yield as the second state variable is relevant. On that point of view, the selection of the two state variables introduced in the short-term/long-term model offers an improvement because the short-term and the long-term deviations are more "orthogonal" in their dynamics (the correlation between these two state variables is lower: it was estimated by Schwartz and Smith at 0.189 and 0.3 for two different datasets). This orthogonality allows distinguishing more clearly the impact of each factor. However, one may also ask if the long-term price, which is not very volatile, is a good candidate for a stochastic variable.

Three-factor models

The empirical results obtained with the three-factor model developed by Schwartz in 1997 raise questions on the relevance of such models. Indeed, using forward prices provided by Enron, the author showed that the two and the three-factor models are empirically very similar. Indeed, these models imply very similar futures volatilities. This result was confirmed by principal component analysis of term structure performed on the copper and the crude oil market.

SECTION 4. APPLICATIONS OF TERM STRUCTURE MODELS

Two important applications have been considered for term structure model of commodity prices: dynamic hedging strategies and investment decisions. These two applications exploit the relationship between futures prices of different maturities.

[15] Schwartz (1997), Lautier and Galli (2001), Lautier (2003).

4.1. Dynamic hedging strategies

The first application of term structure models is the hedge of long-term commodity commitments, when "naïve" hedging strategies are neglected. Naïve strategies suggest taking a short position in the futures market in order to hedge a long position in the physical market, the two positions having the same size and expiration date. These strategies cannot be initiated when the position to hedge has a horizon superior to the exchange-traded maturities. Whereas this kind of problem was previously tackled by Ederington (1979), it acquires a new dimension with term structure models. Indeed, these models rely on arbitrage reasoning and on the construction of a hedge portfolio. Therefore, their elaboration leads naturally to the study of hedging strategies.

Reflection on the use of term structure models for hedging purposes was motivated by Metallgesellchaft's experience. At the beginning of the nineties, this firm tried to cover long-term forward commitments on the physical market with short-term futures contracts. This attempt ended in a resounding failure: USD 2.4 billions were lost. However, it initiated research aiming to know whether this kind of operation can be safely undertaken. The interest of the Metallgesellschaft case is twofold. Firstly, being able to have long-term exposures on the physical market while hedging the product of such sales constitutes an important stake. The success encountered by Metallgesellschaft when it proposed forward sales for a horizon of 5 or 10 years is a good illustration of that point. Secondly, Metallgesellschaft tried to hedge its long-term commitments in the physical market with short-term positions in the futures market. This kind of strategy is particularly interesting when the maturity of actively traded futures contracts is limited to a few months. The use of the nearest maturities supposes however that the hedge portfolio is rebalanced regularly as the future's maturity date approaches, in order to maintain constantly the position on the paper market. Thus, this strategy exposes to a rollover basis risk which, associated with a bad hedging ratio, contributed to the ruin of Metallgesellschaft[16].

The various hedging strategies based on term structure models differ mainly from each other in the assumptions concerning the behaviour of the futures prices. In 1997, Brennan and Crew compared the hedging strategy initiated by Metallgesellschaft on the crude oil market with other strategies relying on Brennan and Schwartz's model (1985) and Gibson and Schwartz's model (1990). The authors showed that the strategies relying on the term structure models outperform largely the one chosen by the German firm all the more, as the term structure model is able to replicate correctly the prices curve empirically observed. The same year, Schwartz also calculated the hedge ratios associated with each of the three models he developed. However, he did not test the efficiency of the related hedging strategies. In 1999, Neuberger compared the performances of hedging strategies relying on the two-factor model developed by Schwartz in 1997 and a new model. In this new model, no assumptions are made on the number of state variables, about the process they follow, or about the way risk is priced. The key assumption is that the expected price at which the long-dated contract starts trading is a linear function of the price of existing contracts (whereas in Schwartz's model, the price of a futures contract can be expressed as a non-linear function of the state variables). While

[16] For more information on the Metallgesellschaft case, see for example Culp and Miller (1994, 1995), Edwards and Canter (1995).

strictly inconsistent with some models of the term structure of commodity prices, this new model still gives good results in practice. Indeed, with this model, the author constructed hedge portfolios based on an arbitrary number of futures contracts with different maturities. His strategy was quite successful in eliminating most of the risk exposure (approximately 85% as measured by the standard deviation of the hedge error) in the crude oil market. On the author's point of view, the robustness of his model derives from the paucity of assumptions. The drawbacks of hedging strategies based on term structure models is that the latter are efficient only if futures prices are fairly priced relative to each others. However, the author underlines that his model requires balancing the portfolio a bit more frequently than Schwartz's one. In 2000, Routledge, Seppi and Spatt showed that hedge ratios based on their term structure model are not constant, but are conditional on the current demand shock and the endogenous inventory level. These ratios take into account the fact that short-term and long-term forward prices differ, in that long-term prices do not include the option to consume the good between the two delivery dates. Lastly, in 2002, Veld-Merkoula and de Roon used a one-factor term structure model based on convenience yield to construct hedge strategies that minimize both spot price risk and rollover risk by using futures of two different maturities. They take into account the transaction costs associated with their strategy, which outperforms the naïve hedging strategy. However, the authors did not compare their results with previous works.

These works on hedging strategies associated with term structure models share some general features. Firstly, the hedging problem is always approached using combinations of futures contracts having different maturities and exploiting the relationship between the futures prices. The number of futures positions is at least equal to the number of underlying factors (*i.e.* state variables) included in the term structure model. Secondly, to hedge properly a forward commitment, the sensitivity of the present value of the commitment with respect to each one of the underlying factors must equal the sensitivity of the portfolio of futures contracts used to hedge the commitment with respect to the same factors. Therefore, the hedge ratios are state dependent and they decline with the maturity of the forward position. Thirdly, until now, the maturities of the futures contracts forming the hedge portfolio are always chosen arbitrarily, like the rebalancing of the portfolio. Lastly, little work has been done on transactions costs and on the financing costs associated with the positions on the futures market.

4.2. Investment decision

The second application of term structure models of commodity prices is the investment decision. The use of term structure models in the case of the investment decision is rather intuitive: with such a model, it is possible to compute a futures price for any expiration date, even if it is very far away. Thus, such a model enables the valuation of the net cash flows associated with an investment project. All the studies using term structure models for the investment decision are conceived in the framework of real options, and the commodities considered are mineral reserves. The real option theory is based on an analogy with the financial options[17]. It aims to identify the optional component included in most investment projects, and to evaluate it, when possible. Its main advantage is that,

[17] A presentation of the real options theory can be found in Copeland and Antikarov (2001), Grinblatt and Titman (2001), Trigeorgis (1999).

contrary to the methods traditionally used for the selection of investment projects - like the net present value - it takes into account the flexibility of a project. This is all the more important as irreversibility is associated with the project – which is the case of most mineral investments.

The theory leads to the identification of different families of real options and underlines that most investment projects include several options. Therefore, the studies realized in the field of commodities take into account various real options. In the beginning, the analysis framework is quite idealistic: everything except the price of the commodity, is supposed to be known. Since 1998 however, other sources of uncertainty are also taken into account.

The pioneer article on investment decision was written by Brennan and Schwartz in 1985. The authors considered a mine, where the resource to be exploited is homogenous and of a known amount, costs are known and interest rates are non-stochastic. There is an upper limit to the output rate, and the study takes into account the possibility of closing and reopening the mine in response to current market conditions. The main source of uncertainty, in that case, is the commodity's price, whose dynamic behaviour is represented with the one-factor term structure model presented below. Thus, in this framework, there are several real options associated with the possession of the mine: the option to shut down the mine temporarily, the abandonment option, and the option to defer investment. The later is the simplest real option and undoubtedly the most frequently evoked in the literature. It represents the possibility to wait before investing in order to collect some useful information. With the parameters values they chose for simulations, the authors find that it is never optimal, under uncertainty, to close or abandon the mine. They also show how the option value changes with the volume of the reserve, with the initial amount to invest, etc.

In 1997, Cortazar and Schwartz use a one-factor model based on mean reverting spot price, in which the convenience yield is variable and depends on the deviation of the spot price to a long-term average price. Using this model, they calculate the value of the field at different stages. Stage one corresponds to the field before committing to the development, stage two is the field during development, and stage three is the field during production. They show that the flexibility to wait before investment can amount 10% to 40% of the field value, and that the timing option is an increasing function of the spot price and a decreasing function of the available time to develop.

The same year, Schwartz shows how the value of a copper mine varies with the underlying term structure models chosen for the valuation. He considers only the option to delay and he computes the trigger price at which it is optimal to invest. The study shows that the value of a real option and the investment decision depend strongly on the method used for the valuation of the net future cash flows associated with an investment project. Indeed, the simulations indicate that the assumptions on the dynamic behaviour of the state variables in the term structure model have a considerable influence on the project's value and on the investment decision.

In 1998, Schwartz develops a one-factor model that retains most of the characteristics of the more complex two-factor model proposed in 1997 in term of its ability to price the term structure of futures prices and volatilities. The model is based on the pricing and volatility results of the two-factor model but, when applied to value long-term commodity projects, it only requires the numerical solution corresponding to a typical one-factor model. The author shows that this one-factor model has practically the same implications as the two-factor model when it is applied to value long-term commodity assets. This one-factor model can be used to value real options that are more complex, without sacrificing any of the advantages of the two-factor model. Still, the two-factor model is needed because the value of its parameters is used as an input for the one-factor model.

The same year, Smith and McCardle propose a model of an oil property where both the price of oil and the production rate vary stochastically over time and at any time, the decision maker has the possibility to terminate or accelerate production by drilling additional wells. The decision maker may also hedge some, but not all of the risks associated with the project.

In 2000, Schwartz and Smith apply their short-term/long-term model to some hypothetical real options problem. They consider two real options: the option to defer investment for a long-term investment, and the development option for a short-term project. To simplify the analysis, they assume that there are no operating costs, royalties and taxes, and that once production starts, it continues indefinitely. The authors show that in the short-term project, the values and policies are sensitive to both state variables and the value increases with both the short-term deviations and the equilibrium price. In contrasts, the value and policies of the long-term project are quite insensitive to the short-term deviations. Therefore, the authors propose, for long-term analysis, to reduce this two-factor model to a one-factor model that considers uncertainty in the equilibrium price only. However, the two-factor model would always be needed to estimate the equilibrium price.

In 2001, Cortazar, Schwartz and Cassassus collapse price and geological-technical uncertainty into a one-factor model. Using this model, they determine the value of several real options: flexible investment schedules for all exploration stages, and a timing option for the development investment. Moreover, once the mine is developed, there are closing, opening and abandonment options for the extraction phase. The model is applied to the copper market. The authors find that a significant portion of the field value is due to the operational, development and exploitation options available to project managers.

All the studies on investment decision share some general features. Firstly, the analysis framework is usually quite simple, because the valuation of an option is generally much more complicated that the valuation of a futures contract. The same reason leads generally to the use of simple term structure models. Most of the time, even one-factor models do not have an analytical solution. Lastly, very poor empirical work has been done in that field, simply because the parameters estimation becomes impossible when the horizon of analysis exceeds the exchange-traded maturities.

The stake and the interest of these studies are beyond the scope of the simple interrogations concerning the optimal exploitation of a field or a mine. In the petroleum industry for example, the exploration phase is indeed particularly risky and the length of the return on investment period is important. Consequently, only private funds can generally be used to finance such projects. However, provided he benefits from reliable methods to valuate a field and to hedge the associated cash flows, a producer could use its guaranteed profits on the field as collateral to banking funds, and thus reach funding that were until then inaccessible.

SECTION 5. CONCLUSION

In this final section, we can identify some of the broad trends in the literature on commodity pricing during the 1990s and early 2000s.

Firstly, considering the main developments on term structure models of commodity prices, it is possible to determine some specificity of commodities that distinguish them from other assets. Commodities are indeed characterized by mean reversion in spot and futures prices. Moreover, because arbitrage relationships between the futures market and the physical market are limited, price volatility is positively correlated with the degree of backwardation. Prices are also sometimes affected by seasonality. Lastly, the term structure is characterized by the Samuelson effect.

Secondly, independently of the number of state variables included, the term structure models of commodity prices are generally conceived in a partial equilibrium framework[18]. Consequently, the selection of the state variables can be considered as somehow arbitrary. However, the choice is most of the time based on the traditional theories. Moreover, autonomous spot price, convenience yield and long-term price may be regarded as the reduced form of a more general model in which these variables are endogenously determined by production, consumption and storage decisions. Still, until now, nobody has really proved that the convenience yield is a better choice than the long-term price as a second factor. The comparison between the models is quite difficult to undertake.

Future developments in term structure model of commodity prices will probably introduce a more precise description of the prices behaviour. Until now, the leptokurtic nature of commodity returns (Dusak, 1973), and the fact that commodity returns distribution are negatively skewed were for example ignored. The introduction of such characteristics in term structure models could lead to an improvement of the performances. However, in that case, the question of the arbitrage between reality and simplicity arises. Although such an introduction may improve the performances of the models, there will be a balance to find between the fidelity of the prices models and the need for parsimony, especially when the models are conceived for the evaluation of more complex derivatives products, as real options.

As far as the applications of term structure models are concerned, almost two directions can be drawn. For hedging purposes, to be adapted by practitioners, the literature could progress towards practical considerations, like the transactions costs associated with hedging portfolios or the rebalancing of these portfolios. Moreover, there is a need to quantify the risk associated with these portfolios, using for example "value at risk" methods (Cabedo and Moya (2003)). For the valuation of real assets relying on the theory of real options, it could be interesting to introduce other sources of uncertainty in the valuation process. Until now, the analysis framework taken into consideration is most of the time simplistic, and the main source of uncertainty is the price of the commodity. However, the introduction of new sources of uncertainty prevents probably from pricing several options simultaneously. Once again, some arbitrages must be done.

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[18] To be exhaustive on term structure models, we must also quote the studies undertaken by Cortazar and Schwartz (1994) and by Miltersen and Schwartz (1998). They were the only one to propose a general framework of pricing commodity futures options using the Heath, Jarrow and Morton (1992) methodology (probabilistic approach). Their model takes the entire term structure of futures prices as given.

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