

# Existence of price functionals with bid-ask spreads on the space of all integrable contingent claims

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**Abstract:** Jouini [Price functionals with bid-ask spreads: an axiomatic approach, *Journal of Mathematical Economics* **34** (2000), 547-558] presented an axiomatization of the existence of an admissible price functional in case that the set of marketed contingent claims consists of all square integrable random variables. The author characterized the absence of arbitrage opportunities by requiring the admissible price functional to be strictly positive.

In this paper, we present a sufficient condition for the existence of an admissible price functional in the presence of bid-ask spreads in the more general situation when the space of marketed contingent claims is equal to the space of all integrable contingent claims. The assumption of strict positivity of an admissible price functional is replaced by the stronger assumption of strict monotonicity with respect to the natural preorder on the space of all integrable contingent claims. Our result generalizes the sufficiency part in the axiomatization proposed by Jouini in the aforementioned paper.

**Keywords:** price functional, bid-ask spreads, no arbitrage assumption, marketed contingent claims.

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# 1 Introduction

The characterization of a price functional with bid-ask spreads has been recently studied by Jouini [2] by using an axiomatic approach. The situation considered by the author is quite general, since the bid and ask price processes are independent. In fact, there exists an important literature concerning replication costs with bid-ask spreads. For example, such a problem has been analyzed by Boyle and Vorst [1] and Koehl, Pham and Touzi [3] amongst others, but the bid and ask price functionals are assumed to be dependent. Therefore, Jouini [2] has considered a more general framework.

Jouini [2] provided a very elegant characterization of the existence of an admissible price functional on the space of all square integrable contingent claims in the presence of bid-ask spreads, where a price functional is said to be admissible if it is sublinear, lower semicontinuous, strictly positive, and it satisfies a very reasonable monotonicity assumption consisting in the requirement that the price of any contingent claim is less than or equal to the infimum cost necessary to obtain at least the final contingent portfolio at the final date. In such an axiomatization, the underlying probability space  $(\Omega, \mathcal{F}, P)$  is arbitrary, and it is only assumed that the space of all square integrable random variables  $L^2(\Omega, \mathcal{F}, P)$  is separable.

In this paper, we provide a simple sufficient condition for the existence of an admissible price functional on the space of all contingent claims which are integrable (i.e., with finite expectation). In the no-arbitrage assumption concerning a price functional, strict positivity is replaced by the stronger hypothesis of strict monotonicity with respect to the natural preorder on the space of marketed contingent claims. Therefore, the sufficiency part in the aforementioned result by Jouini is generalized by our condition.

## 2 The model

In the sequel, we shall consider an arbitrary *probability space*  $(\Omega, \mathcal{F}, P)$ . Given a finite horizon  $T$ , let  $\mathbb{F} = \{\mathcal{F}_t\}_{t=0, \dots, T}$  be a *filtration* (i.e., an increasing family of sub  $\sigma$ -algebras of  $\mathcal{F}$ ). As usual,  $\mathcal{F}_t$  represents the information

available at time  $t$ . We shall assume that  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  and  $\mathcal{F}_T = \mathcal{F}$ .

The market consists of one riskless asset and one risky asset<sup>1</sup>, and we shall denote by  $A^0(t)$  and  $B^0(t)$  the *ask* and respectively the *bid price* at time  $t \in \{0, \dots, T\}$  of the riskless asset, and by  $A^1(t)$  and  $B^1(t)$  the ask and respectively the bid price at time  $t \in \{0, \dots, T\}$  of the risky asset. Therefore, the asset indexed by  $i \in \{0, 1\}$  can be bought for its ask price  $A^i(t)$  and can be sold for its bid price  $B^i(t)$  at any time  $t \in \{0, \dots, T\}$ . The random variables  $A^i(t)$  and  $B^i(t)$  are assumed to be  $\mathcal{F}_t$ -measurable for every  $t \in \{0, \dots, T\}$  and  $i \in \{0, 1\}$ . If we let  $A(t) = (A^0(t), A^1(t))$  and  $B(t) = (B^0(t), B^1(t))$  ( $t \in \{0, \dots, T\}$ ), it must be  $A(t) \geq B(t) > 0$  a.s. for every  $t \in \{0, \dots, T\}$ .

Define  $\tilde{\Omega} = \Omega \times \{0, 1\}$  and  $\tilde{X} = L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ , with  $(\tilde{\mathcal{F}}, \tilde{P})$  the natural probability structure<sup>2</sup> induced by  $(\mathcal{F}, P)$  on  $\tilde{\Omega}$ . Further, consider the preorder (i.e., the reflexive and transitive binary relation)  $\preceq$  on  $\tilde{X}$  defined as follows:

$$x_1 \preceq x_2 \Leftrightarrow \tilde{P}(x_2 \geq x_1) = 1 \quad (x_1, x_2 \in \tilde{X}).$$

If we let  $\tilde{X}_+ = \{x \in \tilde{X} : (\tilde{P}(x \geq \mathbf{0}) = 1) \text{ and } (\tilde{P}(x > \mathbf{0}) > 0)\}$ , then it is easily seen that the *strict part*<sup>3</sup>  $\prec$  of the preorder  $\preceq$  is defined by

$$x_1 \prec x_2 \Leftrightarrow x_2 - x_1 \in \tilde{X}_+ \quad (x_1, x_2 \in \tilde{X}).$$

**Definition 2.1.** A functional  $p$  on  $\tilde{X}$  with values in  $\mathbb{R} \cup \{\infty\}$  is said to be *strictly monotone with respect to the preorder  $\preceq$*  if  $p$  is *monotone with respect to  $\preceq$*  (i.e., for every  $x_1, x_2 \in \tilde{X}$ ,  $p(x_1) \leq p(x_2)$  whenever  $x_1 \preceq x_2$ ), and in addition, for every  $x_1, x_2 \in \tilde{X}$ ,  $p(x_1) < p(x_2)$  whenever  $x_1 \prec x_2$ .  $\square$

In the sequel, a functional on  $\tilde{X}$  which is (strictly) monotone with respect to the preorder  $\preceq$  previously defined will be simply referred to as a (*strictly*) *monotone functional*.

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<sup>1</sup>The assumption that there is only one risky asset is made for the ease of the exposition. We could have considered the situation when there exist  $k \geq 1$  risky assets.

<sup>2</sup>Observe that  $\tilde{\mathcal{F}} = \{F \times \{0, 1\} : F \in \mathcal{F}\}$ , and  $\tilde{P}(F \times \{i\}) = \frac{1}{2}P(F)$  for all  $F \in \mathcal{F}$  and  $i \in \{0, 1\}$ . Hence, a  $\mathbb{R}^2$ -valued random variable  $x = (x^0, x^1)$  on the probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$  belongs to  $\tilde{X}$  if and only if the  $\mathbb{R}$ -valued random variables  $x^0$  and  $x^1$  on  $(\Omega, \mathcal{F}, P)$  belong to the space  $L^1(\Omega, \mathcal{F}, P)$ .

<sup>3</sup>We recall that the strict part  $\prec$  of any preorder  $\preceq$  on  $\tilde{X}$  is defined by  $[x_1 \prec x_2 \Leftrightarrow (x_1 \preceq x_2) \text{ and } \text{not}(x_2 \preceq x_1)]$  ( $x_1, x_2 \in \tilde{X}$ ).

**Definition 2.2.** A *simple strategy*  $(\theta_A, \theta_B)$  is a pair of  $\mathbb{R}^2$ -valued stochastic processes adapted to  $\{\mathcal{F}_t\}_{t=0, \dots, T}$  such that:

- (i)  $\theta_A$  and  $\theta_B$  are nonnegative and non-decreasing processes;
- (ii)  $\theta_A(t) \cdot A(t)$  and  $\theta_B(t) \cdot B(t)$  belong to  $L^1(\Omega, \mathcal{F}_t, P)$  for every  $t \in \{0, \dots, T\}$ .  $\square$

According to definition 2.2, it is apparent that a simple strategy is interpreted as cumulative long and respectively short positions.

**Definition 2.3.** A simple strategy  $(\theta_A, \theta_B)$  is said to be *self-financing* if, for every  $t \in \{1, \dots, T\}$ ,

$$(\theta_A(t) - \theta_A(t-1)) \cdot A(t) \leq (\theta_B(t) - \theta_B(t-1)) \cdot B(t) \quad a.s..$$

$\square$

By definition 2.3, if a simple strategy is self-financing, then purchases must be financed by sells. The set of all simple self-financing strategies will be denoted by  $\Theta$ .

**Definition 2.4.** A *contingent claim*  $C = (C^0, C^1)$  is an element of  $\tilde{X}$  (i.e., a pair of  $\mathcal{F}$ -measurable random variables with finite expectation).  $\square$

A contingent claim  $C = (C^0, C^1)$  represents a time  $T$  portfolio consisting of  $C^0$  units of the riskless asset and  $C^1$  units of the risky asset. Therefore, in our framework,  $\tilde{X}$  is the set of all *marketed* contingent claims which are integrable.

It is well known that in finance a classical example of a contingent claim is provided by European call options. Indeed, given the exercise price  $K$ , the maturity time  $T$  and the price of the underlying risky asset at maturity  $S_T$ , we have  $C^0 = -K \mathbf{1}_{\{S_T \geq K\}}$  and  $C^1 = \mathbf{1}_{\{S_T \geq K\}}$ .

**Definition 2.5.** A price functional  $p$  is a functional on the set  $\tilde{X}$  of marketed contingent claims with values in  $\mathbb{R} \cup \{\infty\}$ .  $\square$

Given any price functional  $p$ , the value  $p(C)$  represents the price at which the marketed contingent claim  $C$  can be bought.

**Definition 2.6.** We say that a price functional  $p$  is *admissible* if it satisfies the following conditions:

- (1)  $p$  is *sublinear*<sup>4</sup>;
- (2)  $p$  is *lower semicontinuous*<sup>5</sup>;
- (3)  $p$  is *strictly monotone*;
- (4)  $p(C) \leq \pi(C) = \inf\{\theta_A(0) \cdot A(0) - \theta_B(0) \cdot B(0) : (\theta_A, \theta_B) \in \Theta \text{ and } (\theta_A - \theta_B)(T) \geq C\}$  for every contingent claim  $C$ .  $\square$

It should be noted that the above conditions (1) and (3) imply that the functional  $p$  induces *no-arbitrage* (i.e.,  $p(C) > 0$  whenever  $C \in \tilde{X}_+$ ). Indeed, if  $p$  is positively homogeneous, then it is necessarily  $p(\mathbf{0}) = 0$ . Therefore, under our definition of an admissible price functional, condition (3) above is stronger than the absence of arbitrage condition proposed by Jouini [2, axiom 3], while conditions (1), (2) and (4) are identical to axioms 1, 2 and 4 in the the same paper of Jouini, who clearly explains that such requirements are all very natural for any price functional. We just recall that, according to the definition of the functional  $\pi$  in condition (4), the value  $\pi(C)$  represents the infimum cost to be sustained in order to obtain at least the final contingent portfolio  $C$  at time  $T$ .

### 3 Existence of admissible price functionals

In this section, we are concerned with the existence of an admissible price functional on the space of all integrable contingent claims. We recall that, given a probability space  $(\Omega, \mathcal{F}, P)$ , a probability measure  $Q$  is said to be *equivalent* to the probability measure  $P$  if  $P$  and  $Q$  have the same

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<sup>4</sup>A functional  $p$  is said to be sublinear if it is *positively homogeneous* and *subadditive*. We recall that a functional  $p$  is said to be positively homogeneous if  $p(\lambda C) = \lambda p(C)$  for every nonnegative real number  $\lambda$  and every element  $C$  of  $\tilde{X}$ ; it is said to be subadditive if  $p(C_1 + C_2) \leq p(C_1) + p(C_2)$  for every pair  $(C_1, C_2)$  of elements of  $\tilde{X}$ . It is clear that a sublinear functional is also convex, i.e.  $p(\lambda C_1 + (1 - \lambda)C_2) \leq \lambda p(C_1) + (1 - \lambda)p(C_2)$  for every pair  $(C_1, C_2)$  of elements of  $\tilde{X}$  and every real number  $\lambda \in [0, 1]$ .

<sup>5</sup>Generally speaking, a functional  $p$  on  $\tilde{X}$  is said to be lower semicontinuous if  $\{C \in \tilde{X} : p(C) \leq \lambda\}$  is a closed subset of  $\tilde{X}$  for every  $\lambda \in \mathbb{R}$ . Since we consider the  $L^1$ -norm, and  $\tilde{X}$  is obviously a metric space, this is equivalent to require that for every sequence  $\{C_n\} \subseteq \tilde{X}$ , element  $C \in \tilde{X}$  and real number  $\lambda$  such that  $C_n \rightarrow C$  as  $n \rightarrow \infty$ , and  $p(C_n) \leq \lambda$  for every integer  $n$ , it is  $p(C) \leq \lambda$ .

zero measure sets (i.e., for all sets  $F \in \mathcal{F}$ ,  $[Q(F) = 0 \Leftrightarrow P(F) = 0]$ ). Further, given a probability space  $(\Omega, \mathcal{F}, P)$  equipped with a filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t=0, \dots, T}$ , a stochastic process  $\{M(t)\}_{t=0, \dots, T}$  adapted to  $\mathbb{F}$  is said to be a *supermartingale* with respect to the probability measure  $P$  and the filtration  $\mathbb{F}$  if, for every  $t \in \{1, \dots, T\}$ ,

$$E^P(M(t) \mid \mathcal{F}_{t-1}) \leq M(t-1).$$

Now we are ready to present the main result of this paper.

**Theorem 3.1.** *A sufficient condition for the existence of at least one admissible price functional  $p$  is that there exist*

- (1) *a family  $\mathcal{Q}$  of probability measures equivalent to  $P$  such that  $dQ/dP \in L^1(\Omega, \mathcal{F}, P)$  for every  $Q \in \mathcal{Q}$ ;*
- (2) *an adapted stochastic process  $\{M(t)\}_{t=0, \dots, T}$  with  $A(t) \geq M(t) \geq B(t)$  a.s. for  $t = 0, \dots, T$*

*such that*

- (\*)  *$(dQ/dP)M(t) \in L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$  for every  $Q \in \mathcal{Q}$ ;*
- (\*\*)  *$\{M(t)\}_{t=0, \dots, T}$  is a supermartingale with respect to the filtration  $\mathbb{F}$  and the probability measure  $Q$  for every  $Q \in \mathcal{Q}$ .*

*Proof.* Define the functional  $p$  on  $\tilde{X}$  by

$$p(C) = \sup\{E^Q(M(T) \cdot C) : Q \in \mathcal{Q}\} \quad (C \in \tilde{X}).$$

We claim that the functional  $p$  is an admissible price functional. For convenience, let us define, for every  $Q \in \mathcal{Q}$ ,

- (i)  $\rho_Q = dQ/dP$ ;
- (ii)  $f_Q : C \rightarrow E^P(\rho_Q M(T) \cdot C)$ .

Then we have

$$p(C) = \sup\{f_Q(C) : Q \in \mathcal{Q}\}.$$

–) Let us first show that  $p$  is sublinear. We observe that  $p$  is positively homogeneous, since the functional  $f_Q$  is positively homogeneous for every  $Q \in \mathcal{Q}$ . In order to show that  $p$  is also subadditive, assume by contradiction that there exist two contingent claims  $C_1, C_2$  such that  $p(C_1) + p(C_2) < p(C_1 + C_2)$ . Then, from the definition of  $p$ , there exists a probability measure  $Q \in \mathcal{Q}$  such that  $p(C_1) + p(C_2) < E^Q(\rho_Q M(T) \cdot (C_1 + C_2)) < p(C_1 + C_2)$ . Hence we have that either  $p(C_1) < f_Q(C_1)$  or  $p(C_2) < f_Q(C_2)$ , and this is contradictory.

–) In order to prove that  $p$  is lower semicontinuous, consider any sequence of contingent claims  $\{C_n\} \subseteq \tilde{X}$ , any contingent claim  $C \in \tilde{X}$  and any real number  $\lambda$  such that  $C_n \rightarrow C$  as  $n \rightarrow \infty$  and  $p(C_n) \leq \lambda$  for every integer  $n$ . Then it must be  $p(C) \leq \lambda$  from the definition of  $p$ , because the functional  $f_Q$  is continuous in  $L^1$ -norm for every  $Q \in \mathcal{Q}$ , and therefore  $f_Q(C) \leq \lambda$  for every  $Q \in \mathcal{Q}$ .

–) Further,  $p$  is strictly monotone since the functional  $f_Q$  is strictly monotone for every  $Q \in \mathcal{Q}$ .

–) It only remains to check that  $p(C) \leq \pi(C)$  for all contingent claims  $C$ . Consider any contingent claim  $C$  and any simple self-financing strategy  $(\theta_A, \theta_B)$  such that  $(\theta_A - \theta_B)(T) \geq C$ . Using the fact that  $(\theta_A, \theta_B)$  is also non-decreasing, and that  $A(t) \geq M(t) \geq B(t)$  for  $t = 0, \dots, T$ , it is easily seen that, for every  $Q \in \mathcal{Q}$  and  $t \in \{1, \dots, T\}$ , we have

$$\begin{aligned} E^Q((\theta_A(t) - \theta_A(t-1)) \cdot M(t) - (\theta_B(t) - \theta_B(t-1)) \cdot M(t) \mid \mathcal{F}_{t-1}) &\leq \\ E^Q((\theta_A(t) - \theta_A(t-1)) \cdot A(t) - (\theta_B(t) - \theta_B(t-1)) \cdot B(t) \mid \mathcal{F}_{t-1}) &\leq 0. \end{aligned}$$

Since  $\{M(t)\}_{t=0, \dots, T}$  is a supermartingale with respect to the filtration  $\mathbb{F}$  and the probability measure  $Q$  for every  $Q \in \mathcal{Q}$ , we have

$$\begin{aligned} E^Q((\theta_A - \theta_B)(t) \cdot M(t) \mid \mathcal{F}_{t-1}) &\leq E^Q((\theta_A - \theta_B)(t-1) \cdot M(t) \mid \mathcal{F}_{t-1}) \leq \\ &(\theta_A - \theta_B)(t-1) \cdot M(t-1), \end{aligned}$$

which implies by iteration

$$\begin{aligned} E^Q((\theta_A - \theta_B)(T) \cdot M(T)) &\leq (\theta_A - \theta_B)(0) \cdot M(0) \\ &\leq \theta_A(0) \cdot A(0) - \theta_B(0) \cdot B(0). \end{aligned}$$

Since we have  $E^Q(M(T) \cdot C) \leq E^Q(M(T) \cdot (\theta_A - \theta_B)(T))$  for every  $Q \in \mathcal{Q}$ , it is also  $p(C) \leq \theta_A(0)A(0) - \theta_B(0)B(0)$  from the definition of  $p$  for every strategy  $(\theta_A, \theta_B)$  satisfying the previous requirements. Hence, it is  $p(C) \leq \pi(C)$ . Such consideration completes the proof.  $\square$

**Remark 3.1.** Since it is clear that the space  $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$  is contained in the space  $L^1(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ , the above theorem 3.1 generalizes the sufficiency part of Theorem 1 in Jouini [2], who requires the existence of one probability measure  $Q$  equivalent to  $P$ , and one martingale  $\{M(t)\}_{t=0, \dots, T}$  with respect to the filtration  $\mathbb{F}$  and the probability measure  $Q$  such that  $A(t) \geq M(t) \geq B(t)$  a.s. and  $dQ/dP \in L^2(\Omega, \mathcal{F}, P)$ .  $\square$

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