DORNBUSCH OR FRANKEL ?: THE SAME MODEL

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I) INTRODUCTION

It is generally acknowledged that the Dornbusch[2] sticky price monetarist model — also called overshooting model — represents a major contribution to exchange rate theory in the sense that departures from purchasing power parity and volatile currency levels can be explained rationally.

Conversely, it is also advocated that the model exhibits several shortcomings. H. Bourguinat[3] quotes that a weighted average of domestic and import prices would have been a more appropriate deflator but would have weakened Dornbusch's results. He also points out that domestic agents don't hold external money balances, which limits the scope of the model. He asserts that agents are more and more able to foresee monetary shocks and adjust accordingly their understanding of the long run monetary path. But the most frequent critique is that Dornbusch's model doens't properly handle inflation. In a classic article, J. Frankel[4] combines sticky prices with secular rates of inflation, and finds that the exchange rate between two countries is driven by their real interest rates differential, and not by their nominal one as in Dornbusch.

The purpose of this article is to show that Dornbusch's and Frankel's models are in fact identical. Consequently, they bring the same results and

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[2] ''Expectations and Exchange Rate Dynamics'', Rudiger Dornbusch, Journal of Political Economy, Volume 84, n°6, 1976.

[3] ''Finance internationale'', Henri Bourguinat, Thémis-PUF, May 1992. See also ''International Finance'', Keith Pilbeam, Macmillan, 1992, for a discussion about the Dornbusch model.

bear the same drawbacks. Part II sets up the overshooting model in a secular inflationary dual framework by relaxing the two original assumptions of constant prices and small country. Part III shows that the exchange rate disequilibrium depends on the same real interest rate differential as with Frankel. Part IV points out the main theoretical pitfall met by both models.

II) THE MODEL

The model we develop here draws much on the sticky price model. But unlike Dornbusch, we don't assume that the domestic country is small, so that the rest of the world, represented as a foreign country, faces comparable money and goods markets. Furthermore, we will not suppose that the exogeneous variables are constant. However investors will still be indifferent between home and abroad in their money lending or borrowing because they are supposed neutral with respect to the risk of currency depreciation.

II.1 Uncovered Interest Parity

Capital is perfectly mobile so that investors can instantly alter the composition of their international investments. Besides they regard domestic and foreign interest bearing assets, that we will name bonds, as equally risky. Because in our simple framework there is one horizon length, one national rate and one national issuer, equal riskiness boils down to investors' risk neutrality with respect to exchange rate volatility. Hence we can say that domestic and foreign bonds are perfect substitutes. This yields the following relationship:

$$r = r^* + E(x) \qquad (1)$$

where r and r* are the domestic and foreign nominal interest rates, and E(x) is the expected depreciation rate of the domestic currency.

If e is the log of the exchange rate of the domestic currency, i.e. the domestic price of one unit of foreign currency, E(x) can be expressed as

[4] ''On the Mark: A Theory of Floating Exchange Rates Based on Real Interest Differentials'', Jeffrey A. Frankel, The American Economic Review, September 1979.

$$E(x) = E(\frac{de}{dt}) = E(\dot{e})$$

provided that the interest rates are instantaneous.

(1) is known as the Uncovered Interest Parity (UIP), and derives its qualifier *uncovered* from the fact that risk neutral investors don't hedge — cover — their foreign belongings.

UIP holds continuously in the model.

II.2 Purchasing Power Parity Departure and Expectations Formation

Contrary to UIP, the Purchasing Power Parity (PPP) holds only in the long run. It doesn't in the short because, for example after a monetary shock, prices and wages adjust much slower than the exchange rate.

Hence we first need to posit (absolute) PPP in the long run:

$$\bar{e} = \bar{p} - \bar{p}^* \qquad \text{(PPP)}$$

where $\bar{e}$, $\bar{p}$ and $\bar{p}^*$ are the logs of the equilibrium spot rate, price at home and price abroad. Note that $\bar{e}$, $\bar{p}$ and $\bar{p}^*$ are not held constant as in Dornbusch, and are implicitly indexed with time, i.e. $\bar{e}_t$, $\bar{p}_t$ and $\bar{p}^*_t$.

Differentiating (PPP) provides:

$$\bar{x} = \Pi - \Pi^*$$
(PPP')

where $\bar{x} = \dot{\bar{e}}$, $\Pi = \dot{\bar{p}}$, and $\Pi^* = \dot{\bar{p}}^*$. This means that the currency depreciation is equal to the inflation differential.

Although $\bar{x}$, Π and Π^* need not be constant, it may be useful to look at the stationary case where $\bar{e}$, $\bar{p}$ and $\bar{p}^*$ evolve at constant growth rates to better understand how they can be non constant in the long run.

Now suppose that for some reason PPP is violated. We can measure the departure from PPP as:

$$e - p + p^*$$

which is frequently called the PPP gap or real exchange rate or else terms of trade, and that can be expressed with the variables' rates of increase:

$$x - \Pi + \Pi^*$$

We postulate that in the long run the economy is stable and PPP gap vanishes through time under price pressure from imported goods.

We then relate the short term track of the exchange rate to the anticipated closure of PPP gap:

$$E(x - \Pi + \Pi^*) = \theta(\bar{e} - e) \quad (2)$$

The larger the departure from PPP, the wider the spread between spot and equilibrium rates — which we name EER gap in this paper[5] — . θ must be positive for stability concern, for example home currency must be cheap to offset an excess domestic inflation[6].

Because in the long-run agents perfectly understand the economy, $E(\Pi) = \Pi$ and $E(\Pi^*) = \Pi^*$, and (2) becomes

$$E(x) = \Pi - \Pi^* + \theta(\bar{e} - e) \quad (2')$$

We can read this relationship in terms of expectations formation. Agents expect the home currency depreciation to fit a long term trend plus any corrective path due to some disturbance. The long term trend obtains when $e = \bar{e}$ and goes with the perfect foresight $E(\bar{x}) = \bar{x}$:

$$E(\bar{x}) = \bar{x} = \Pi - \Pi^* \quad (2'')$$

(2'') is simply the long term (PPP).

Note that in the long range we can write (UIP) as:

$$\bar{r} = \bar{r}^* + \bar{x} \quad (\bar{1})$$

which combined with (2'') provides the real interest rate parity (RIP) in the long-run:

$$\bar{r} - \Pi = \bar{r}^* - \Pi^* \quad (RIP)$$

Combining (1) and (2') gives the equation:

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[5] EER stands for Equilibrium Exchange Rate. The EER achieves internal and external balances. More refined versions are developed by John Williamson, ''Estimates of FEERs'', Estimating Equilibrium Exchange Rates, Institute for International Economics, 1994, and Michael J. Artis & Mark P. Taylor, ''DEER Hunting: Misalignment, Debt Accumulation, and Desired Equilibrium Exchange Rates'', Working Paper, IMF, 1993.

[6] (2) can be rewritten: $E(x) = \theta(\bar{e} - e) + E(\Pi) - E(\Pi^*)$ which is exactly what Frankel proposes in his article.

$$e - \bar{e} = -\frac{1}{?}[(r - \Pi) - (r^* - \Pi^*)] \qquad (3)$$

from which Frankel's work has been labelled ''real interest differential model''.

Note however that $r - \Pi$ is not exactly the real interest rate since Π is not the expected current inflation $E(\dot{p})$ but is the underlying long run inflation. We will address this issue further down in this paper.

II.3 The Money Market

The domestic demand to hold real money balance is given continuously by the conventional logarithmic function

$$\phi y - \lambda r$$

which, equated to the real money supply from the central bank, yields:

$$m - p = \phi y - \lambda r \qquad (4)$$

where m, p and y denote the logs of respectively the nominal money issued by the central bank, the price level and real income or output. (4) is a classic (LM) identity.

Domestic demand is supposed independent of the foreign demand for home money to settle imports.

Here again, yet exogeneous variables, m and y are nevertheless implicitly indexed with time for they are not held constant. For example, we can imagine a case where long run inflation Π is fed by a steady monetary growth of $g_M = dm/dt$. As m and y are exogenous, their long run and short term values are identical, so we leave them without upbar. Conversely, p and r are endogenous and deserves their short- and long-term representations.

In the long range, (4) becomes:

$$m - \bar{p} = \phi y - \lambda \bar{r} \qquad (\bar{4})$$

The same money function applies abroad:

$$m^* - p^* = \phi y^* - \lambda r^* \qquad (4^*)$$

$$m^* - \bar{p}^* = \phi y^* - \lambda \bar{r}^* \qquad (\bar{4}^*)$$

Subtracting (4) from $(\bar{4})$, and (4^*) from $(\bar{4}^*)$, brings two equalities:

$$\lambda(r - \bar{r}) = p - \bar{p}$$

$$\lambda(r^* - \bar{r}^*) = p^* - \bar{p}^*$$

whose difference gives:

$$\lambda[r - \bar{r} - (r^* - \bar{r}^*)] = p - \bar{p} - (p^* - \bar{p}^*) \quad (5)$$

Replacing $\bar{r}^* - \bar{r}$ by $\Pi - \Pi^*$ in the above equality yields:

$$\lambda[r - \Pi - (r^* - \Pi^*)] = p - \bar{p} - (p^* - \bar{p}^*)$$

and with (3) we gets:

$$e = \bar{e} - \frac{1}{\lambda?}[p - \bar{p} - (p^* - \bar{p}^*)] \quad (6)$$

This is a key equation as it relates the exchange rate to the relative price level departure from its long run trend.

II.4 The Goods Market

The function depicting the demand for domestic goods in an open economy is classic:

$$d = u + \gamma y - \sigma r + \delta(e + p^* - p) \quad (7)$$

where d is the logarithm of the demand D and u embodies the autonomous spending.

At equilibrium supply Y meets demand D, which is the (IS) basics. But reaching equilibrium takes time in our model where prices are sticky. Hence after a disturbance, (7) is violated in the short run and generates an excess demand. We suppose that excess demand D – Y gives rise to an instantaneous short-term inflation $\dot{p}$ in the following way:

$$\dot{p} = \pi \ln(D/Y) = \pi[u + (\gamma - 1)y - \sigma r + \delta(e + p^* - p)] \quad (8)$$

At equilibrium, we get $\dot{\bar{p}} = \Pi$ and hence

$$\Pi = \pi[u + (\gamma - 1)y - \sigma\bar{r}] \quad (\bar{8})$$

so we can rewrite (8) as

$$\dot{p} = \delta\pi(e + p^* - p) + \Pi - \sigma\pi(r - \bar{r}) \quad (8')$$

Price disequilibrium depends on both PPP gap and interest rate disequilibrium. This results differs from Frankel's assumption where excess inflation is related solely to PPP gap.

Now the same demand function prevails abroad and so we get similar identities:

$$\dot{p}^* = \pi[u + (\gamma - 1)y^* - \sigma r^* + \delta(p - e - p^*)] \qquad (8^*)$$

$$\Pi^* = \pi[u + (\gamma - 1)y^* - \sigma\bar{r}^*] \qquad (\bar{8}^*)$$

$$\dot{p}^* = -\delta\pi(e + p^* - p) + \Pi^* - \sigma\pi(r^* - \bar{r}^*) \qquad (8^{*\prime})$$

Subtracting (8*') from (8) provides:

$$\dot{p} - \dot{p}^* = 2\delta\pi(e + p^* - p) + \Pi - \Pi^* - \sigma\pi[r - \bar{r} - (r^* - \bar{r}^*)] \qquad (8^{*\prime\prime})$$

which with (5) becomes:

$$\dot{p} - \dot{p}^* = 2\delta\pi(e + p^* - p) + \Pi - \Pi^* - \frac{\sigma\pi}{\lambda}[p - \bar{p} - (p^* - \bar{p}^*)]$$

$$\dot{p} - \dot{\bar{p}} - (\dot{p}^* - \dot{\bar{p}}^*) = 2\delta\pi(e + p^* - p) - \frac{\sigma\pi}{\lambda}[p - \bar{p} - (p^* - \bar{p}^*)]$$

$$\dot{\overbrace{p - \bar{p} - (p^* - \bar{p}^*)}} = 2\delta\pi(e + p^* - p) - \frac{\sigma\pi}{\lambda}[p - \bar{p} - (p^* - \bar{p}^*)] \qquad (9)$$

We need to relate the PPP gap to the relative price disequilibrium to get a simple differential equation. This can be achieved through a little algebra. Subtracting (4*) from (4) brings:

$$m - m^* = p - p^* + \phi(y - y^*) - \lambda(r - r^*)$$

and in the long run:

$$m - m^* = \bar{p} - \bar{p}^* + \phi(y - y^*) - \lambda(\bar{r} - \bar{r}^*)$$

and by subtracting the latter from the former:

$$\bar{p} - \bar{p}^* = p - p^* - \lambda[(r - r^* - (\bar{r} - \bar{r}^*)]$$

Using (PPP) and (RIP) in order to substitute $\bar{p} - \bar{p}^*$ with $\bar{e}$ and $\bar{r} - \bar{r}^*$ with $\Pi - \Pi^*$, we get:

$$\bar{e} + p^* - p = -\lambda[(r - \Pi) - (r^* - \Pi^*)]$$

Dividing the two hand sides of this equation with the respective hand sides of (3) yields:

$$\frac{\bar{e} + p^* - p}{e - \bar{e}} = \lambda\theta$$

or equivalently

$$e + p^* - p = (1 + \lambda\theta)(e - \bar{e}) \qquad (10)$$

which with (6) gives the promised relation:

$$e + p^* - p = -\frac{1+\lambda?}{\lambda?}[p - \bar{p} - (p^* - \bar{p}^*)] \qquad (10')$$

Replacing PPP gap in (9) with its relative price departure equivalent, we obtain a simple differential equation:

$$\dot{\overbrace{p - \bar{p} - (p^* - \bar{p}^*)}} = -\upsilon\,[p - \bar{p} - (p^* - \bar{p}^*)] \qquad (11)$$

where

$$\upsilon = \frac{2pd(1+??)+sp?}{??} \qquad (12)$$

and whose solution is:

$$p - p^* = \bar{p} - \bar{p}^* + [p_0 - p_0^* - (\bar{p}_0 - \bar{p}_0^*)]e^{-\upsilon t} \qquad (13)$$

υ can be interpreted as a rate of convergence of the relative price towards its equilibrium path.

II.5 Consistency Condition

(6) shows that the currency follows the same type of convergence:

$$e = \bar{e} - \frac{1}{\lambda?}[p_0 - p_0^* - (\bar{p}_0 - \bar{p}_0^*)]e^{-\upsilon t}$$

$$e = \bar{e} + \frac{1}{\lambda?}(e_0 - \bar{e}_0)e^{-\upsilon t} \qquad (14)$$

Home currency depreciates to catch up its long-term path as long as the price differential is above its equilibrium level.

Taking the time derivative of (14) — $\dot{e} = \bar{e} - \upsilon/\lambda\theta(e_0 - \bar{e}_0)e^{-\upsilon t}$ — , recalling that $\dot{e} = x$ and $\dot{\bar{e}} = \bar{x}$, and using (PPP'), we end up with:

$$x = \Pi - \Pi^* - \upsilon(e - \bar{e})$$

Comparing the latter equality with expectation path description (2') entails that expectations fit realizations only when θ equals υ.

Consistent and stable expectations are then reached when θ verifies:

$$\theta = \frac{2pd(1+??)+sp?}{??} \text{ and } \theta > 0 \qquad (15)$$

Solving the quadratic equation gives:

$$\theta = \frac{p}{2}\left(\frac{\sigma}{\lambda}+2\delta\right) + \left[\frac{p^2}{4}\left(\frac{\sigma}{\lambda}+2\delta\right)^2+\pi\frac{2\delta}{\lambda}\right]^{½} \qquad (16)$$

The only difference with Dornbusch's result is that δ is replaced by 2δ, which is the effect of our two-country framework. It is worth noting that relaxing the hypothesis of constant equilibrium values has no incidence there.

When expectations meet realizations, it means that economic agents have a perfect foresight. This strong assumption reduces the attractiveness of the model as it doesn't show very much realistic. A more appealing model would allow some error in predictions, but without persistent bias relative to outcomes. Agents would then feed rational expectations. To our knowledge, no stochastic setting of the Dornbusch or Frankel's models has been proposed so far[7].

Last, if (16) is a necessary condition for model consistency, nothing tells us that it is also sufficient. We will see a complete discussion about this issue in Part IV.

II.6 Conclusion

We have generalized Dornbusch's overshooting model by relaxing the hypotheses of a small country with constant secular macroeconomic variables, and by assuming a force pulling back the exchange rate towards not a given level but a dynamic equilibrium trajectory. Yet we have found the same kind of behaviour for the relative price and the exchange rate, that is to say an exponentional convergence (in log terms) towards the long-run underlying equilibrium path.

However we will see in Part IV that Dornbusch's original model is not so robust when facing a more realistic stance.

III) Dornbusch's Implied Real Interest Rates Differential

The real interest differential model shown in:

$$e - \bar{e} = -\frac{1}{?}[(r - \Pi) - (r^* - \Pi^*)] \qquad (3)$$

may not deserve its name since the real rates are not computed with the actual inflations $\dot{p}$ and $\dot{p}^*$, but with the underlying long-term inflations Π and Π^*.

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[7] Frankel only hints at it in ''Monetary and Portfolio Balance Models of Exchange Rate Determination'', Economic Interdepedence and Flexible Exchange, MIT Press, 1984.

Nevertheless after handling out a set of equations Frankel derives from his model that the equilibrium exchange rate departure — or EER gap — is linearly dependent on the real interest differential involving current inflations.

We will see below that Dornbusch's generalized model leads to the same kind of relationship. We warn the reader that the algebra involved is simple but somewhat cumbersome.

III.1 Linking the Real Rates Differential to the EER and PPP Gaps

Our first computational objective is to gather the EER gap, PPP gap and real interest differential into a single identity.

First recall the long-term (RIP) :

$$\bar{r} - \Pi = \bar{r}^* - \Pi^* \quad \text{(RIP)}$$

and combine it with (3) to get:

$$e - \bar{e} = -\frac{1}{\theta}[(r - r^*) - (\Pi - \Pi^*)] \quad (3')$$

Plugging (3') into (10) yields:

$$e + p^* - p = -\frac{1+\lambda\theta}{\lambda\theta}[(r - r^*) - (\Pi - \Pi^*)] \quad (17)$$

that is, PPP gap is linearly related to the pseudo-real rates difference.

Now use again (RIP) with equation (8*'') by subsituting $\bar{r} - \bar{r}^*$ with $\Pi - \Pi^*$:

$$\dot{p} - \dot{p}^* = 2\delta\pi(e + p^* - p) + \Pi - \Pi^* - \sigma\pi[(r - r^*) - (\Pi - \Pi^*)] \quad (8^{*''})$$

and use (3') to express $\Pi - \Pi^*$ without the pseudo-real rates spread:

$$\Pi - \Pi^* = \dot{p} - \dot{p}^* - 2\delta\pi(e + p^* - p) - \sigma\pi\theta(e - \bar{e}) \quad (18)$$

Then replace this equivalent value of $\Pi - \Pi^*$ in (18) :

$$e + p^* - p = -\frac{1+\lambda\theta}{\lambda\theta}[(r - \dot{p}) - (r^* - \dot{p}^*) + 2\delta\pi(e + p^* - p) + \sigma\pi\theta(e - \bar{e})]$$

Collecting PPP gap terms gives:

$$e + p^* - p = -\frac{1+\lambda\theta}{\theta + 2\pi(1+\lambda\theta)}[(r - \dot{p}) - (r^* - \dot{p}^*) + \sigma\pi\theta(e - \bar{e})] \quad (19)$$

This is the first identity we were looking for. Frankel finds a similar result but without the $\sigma\pi\theta(e - \bar{e})$ term. This difference stems from his simpler assumption about the goods market. Indeed Frankel's pull back forces follow the process:

$$\dot{p} = \pi\delta(e + p^* - p) + \Pi$$

which is simpler than (8) since current inflation doesn't depend directly on the endogenous current interest rate, contrary to what we have allowed[8].

III.2 The Real Interest Rate Differential at Work

Eliminating $\Pi - \Pi^*$ from (3') by using its value from (18), we find:

$$e - \bar{e} = -\frac{1}{\theta}[(r - r^*) - (\dot{p} - \dot{p}^*) - 2\delta\pi(e + p^* - p) - \sigma\pi\theta(e - \bar{e})]$$

and replacing the PPP gap by its value from (19) brings, after rearranging the real rate differential and the EER gap terms in the right hand side:

$$e - \bar{e} = -\frac{1}{\theta}\left[\frac{\theta}{\theta + 2\pi(1+\lambda\theta)}[(r - \dot{p}) - (r^* - \dot{p}^*)] - \frac{\pi\sigma\theta^2}{\theta + 2\pi(1+\lambda\theta)}(e - \bar{e})\right]$$

Finally, we get the linear relationship[9]:

$$e - \bar{e} = -\frac{1}{\theta(1+\pi\sigma) + 2\delta\pi(1+\lambda\theta)}[(r - \dot{p}) - (r^* - \dot{p}^*)]$$

Frankel's similar relationship coefficient is equal to $1/[\theta+2\delta\pi(1+\lambda\theta)]$ and clearly adding up $(1+\pi\sigma)$ reflects our taking account of the current interest rate in the goods market short-term dynamics.

The main finding in Part III is that the enlarged Dornbusch model delivers one of the main result from Frankel, i.e. that the real interest rates differential drives the currency spot.

IV) STABILITY DISCUSSION

A major weakness in our approach is that it has injected a ad hoc link (2) between PPP gap and EER gap into a classic IS-LM model in an open eco-

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[8] Π involves long-run $\bar{r}$ — see ($\bar{8}$) — not instant r.

[9] Notice that $(r - \dot{p})$ is not per say the real interest rate since $\dot{p}$ is not an inflation expectation but a realized figure. But remind that we are looking for a stable path for the economy and that this requires a perfect foresight from the agents. Hence $\dot{p} = E(\dot{p})$.

nomy and with perfectly mobile capital[10], without checking whether the model is by itself sufficient to provide such a link or an alternative one.

In what follows we get a sense of our model from a Control Science perspective. Without supposing that the expectation formation (2) holds ex ante, we will analyze how an economy whose agents have perfect foresight can be stable.

IV.1 The State and Control Variables of the Model

We can think of the long-term variables, depicted with an upbar, as being the engine of the world economy and as well as a planner constraining it to stick to some route. The short-term variables, they, then describe how the economy reacts to once-and-for-all external shocks.

The economy will be considered as a system described by a subset of non-redundant short-term variables — termed the state variables — and forced by fundamental variables — termed the control variables — to follow some trajectory.

State variables are to be drawn from equations (1), (4) & (4*), (8) & (8*), and from the perfect foresight assumption, all holding continuously. In general for a given problem, the appropriate definition of the state is not unique, there being alternative ways of completely describing the current position of the system.

We collect (1), (4) & (4*), and (8) & (8*) to set up the system:

$$\begin{cases} r = r^* + E(x) & (1) \\ m - p = \phi y - \lambda r & (4) \\ m^* - p^* = \phi y^* - \lambda r^* & (4^*) \\ \dot{p} = \pi[u + (\gamma - 1)y - \sigma r + \delta(e + p^* - p)] & (8) \\ \dot{p}^* = \pi[u + (\gamma - 1)y^* - \sigma r^* + \delta(p - e - p^*)] & (8^*) \end{cases}$$

and recall that perfect foresight means that $E(x) = x$.

The vector composed of $(r\ ;\ r^*\ ;p\ ;p^*\ ;e)$ is a straightforward candidate to become state variables, but by first collapsing the above system we will

[10] See for example the Mundell-Fleming model.

then solve it more easily. We observe that the system can be expressed in terms of current exchange rate and price relative by eliminating the nominal interest rate differential. Indeed:

(8) – (8*), (4) – (4*) and (1) yield respectively:

$$\begin{cases} (\dot{p} - \dot{p}^*) = \pi[(\gamma - 1)(y - y^*) - \sigma(r - r^*) + 2\delta(e + p^* - p)] & (20) \\ (m - m^*) - (p - p^*) = \phi(y - y^*) - \lambda(r - r^*) & (21) \\ (r - r^*) = \dot{e} & (22) \end{cases}$$

(21) allows to express r – r* in terms of prices and exogenous variables:

$$r - r^* = \frac{1}{\lambda}\left[(p - p^*) - (m - m^*) + \phi(y - y^*)\right]$$

which, when replaced in (22) and (20) provides:

$$\begin{cases} \dot{e} = \frac{1}{\lambda}(p - p^*) + \frac{\phi}{\lambda}(y - y^*) - \frac{1}{\lambda}(m - m^*) & (23) \\ \dot{\overbrace{p - p}}{}^* = 2\delta\pi e - (\frac{\sigma}{\lambda} + 2\delta)\pi(p - p^*) + (\gamma - 1 - \frac{\sigma\phi}{\lambda})\pi(y - y^*) + \frac{\sigma\pi}{\lambda}(m - m^*) & (24) \end{cases}$$

Now restate this set of two equations in the long-run:

$$\begin{cases} \dot{\bar{e}} = \frac{1}{\lambda}(\bar{p} - \bar{p}^*) + \frac{\phi}{\lambda}(y - y^*) - \frac{1}{\lambda}(m - m^*) \\ \dot{\overbrace{\bar{p} - \bar{p}}}{}^* = 2\delta\pi\bar{e} - (\frac{\sigma}{\lambda} + 2\delta)\pi(\bar{p} - \bar{p}^*) + (\gamma - 1 - \frac{\sigma\phi}{\lambda})\pi(y - y^*) + \frac{\sigma\pi}{\lambda}(m - m^*) \end{cases}$$

Subtracting the equations in underlying long-run variables from the corresponding equations in current variables gives:

$$\begin{cases} \dot{\overbrace{e - \bar{e}}} = \frac{1}{\lambda}[p - \bar{p} + (p^* - \bar{p}^*)] \\ \dot{\overbrace{p - \bar{p} - (p^* - \bar{p}^*)}} = 2\delta\pi(e - \bar{e}) - (\frac{\sigma}{\lambda} + 2\delta)\pi[p - \bar{p} - (p^* - \bar{p}^*)] \end{cases}$$

This is a system that can be represented in matrix form:

$$\dot{X}(t) = AX(t) \qquad (25)$$

where:

$$X(t) = \begin{bmatrix} e - \bar{e} \\ p - \bar{p} - (p^* - \bar{p}^*) \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1/\lambda \\ 2\delta\pi & -(\sigma/\lambda + 2\delta)\pi \end{bmatrix}$$

X(t) is the vector of state variables and A is a time invariant matrix. (25) is also called the state equation. Notice that X(t) is a vector of gap variables. We now turn to solving the equation.

IV.2 Solving for the State Equation

The state equation reveals a linear continuous-time system free of constaint and whose state variables can be readily expressed as functions of time[11]. The solution of (25) is:

$$X(t) = e^{At}X(0)$$

e^{At} can be computed with the Sylvester method which consists in setting the following determinant equal to zero:

$$\begin{vmatrix} 1 & -?_1 & e^{-\theta_1 t} \\ 1 & -?_2 & e^{-\theta_2 t} \\ I & A & e^{At} \end{vmatrix} \qquad (26)$$

where $-\theta_1$ and $-\theta_2$ are the eigenvalues of matrix A, thus satisfying:

$$\begin{vmatrix} -\theta & -1/\lambda \\ -2\delta\pi & -\theta+(\sigma/\lambda+2\delta)\pi \end{vmatrix} = 0 \qquad (27)$$

(27) leads to the same equation as (15) whose roots are:

$$\theta_1 = \frac{p}{2}\left(\frac{\sigma}{\lambda}+2\delta\right) + \left[\frac{p^2}{4}\left(\frac{\sigma}{\lambda}+2\delta\right)^2+\pi\frac{2\delta}{\lambda}\right]^{1/2} \quad \text{and} \quad \theta_2 = \frac{p}{2}\left(\frac{\sigma}{\lambda}+2\delta\right) - \left[\frac{p^2}{4}\left(\frac{\sigma}{\lambda}+2\delta\right)^2+\pi\frac{2\delta}{\lambda}\right]^{1/2}$$

Hence (26) yields:

$$(-\theta_1 e^{-\theta_2 t} + \theta_2 e^{-\theta_1 t})I - (e^{-\theta_2 t} - e^{-\theta_1 t})A + (-\theta_2 + \theta_1)e^{At} = 0$$

and therefore we have:

$$e^{At} = \frac{1}{?_2 - ?_1}\begin{bmatrix} -\theta_1 e^{-\theta_2 t}+\theta_2 e^{-\theta_1 t} & -\frac{1}{\lambda}(e^{-\theta_2 t}-e^{-\theta_1 t}) \\ -2\delta\pi(e^{-\theta_2 t}-e^{-\theta_1 t}) & -\theta_1 e^{-\theta_2 t}+\theta_2 e^{-\theta_1 t} + (\frac{\sigma}{\lambda}+2\delta)\pi(e^{-\theta_2 t}-e^{-\theta_1 t}) \end{bmatrix}$$

The free state solution is then of the form:

$$e - \bar{e} = a_1 e^{-\theta_1 t} + a_2 e^{-\theta_2 t} \qquad \text{and} \qquad p - \bar{p} - (p^* - \bar{p}^*) = b_1 e^{-\theta_1 t} + b_2 e^{-\theta_2 t}$$

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[11] ''Théorie de la commande temporelle'', C.A. Darmon, Ecole Supérieure d'Electricité, 1980.

where a_1, a_2, b_1, b_2 are constants whose values depend on θ_1, θ_2, $e_0 - \bar{e}_0$, $p_0 - \bar{p}_0 - (p_0^* - \bar{p}_0^*)$ and the structural parameters δ, λ, π, σ:

$$\begin{cases} a_1 = \frac{1}{?_2 - ?_1}[\theta_2(e_0 - \bar{e}_0) + \frac{1}{\lambda}(p_0 - \bar{p}_0 - (p_0^* - \bar{p}_0^*))] \\ a_2 = \frac{1}{?_2 - ?_1}[-\theta_1(e_0 - \bar{e}_0) - \frac{1}{\lambda}(p_0 - \bar{p}_0 - (p_0^* - \bar{p}_0^*))] \\ b_1 = \frac{1}{?_2 - ?_1}[2\delta\pi(e_0 - \bar{e}_0) + (\theta_2 - (\frac{\sigma}{\lambda} + 2\delta)\pi)(p_0 - \bar{p}_0 - (p_0^* - \bar{p}_0^*))] \\ b_2 = \frac{1}{?_2 - ?_1}[-2\delta\pi(e_0 - \bar{e}_0) + (-\theta_1 + (\frac{\sigma}{\lambda} + 2\delta)\pi)(p_0 - \bar{p}_0 - (p_0^* - \bar{p}_0^*))] \end{cases}$$

It is clear that the economy is stable if and only if $a_2 = b_2 = 0$. This condition summarizes as:

$$\begin{cases} e_0 = \bar{e}_0 - \frac{1}{\lambda ?_1}[(p_0 - \bar{p}_0 - (p_0^* - \bar{p}_0^*))] \\ e_0 = \bar{e}_0 - \frac{1}{2\delta\pi}(\theta_1 - (\frac{\sigma}{\lambda} + 2\delta)\pi)[(p_0 - \bar{p}_0 - (p_0^* - \bar{p}_0^*))] \end{cases} \quad (28)$$

The two equations collapse into one because we know from (27) that θ_1 verifies $\frac{1}{\lambda ?_1} = \frac{1}{2\delta\pi}(\theta_1 - (\frac{\sigma}{\lambda} + 2\delta)\pi)$.

Let us keep the first one (28). We notice that this is identity (6) at time t = 0. It is easy to see that having (6) at time t = 0 is equivalent to having (2) at time t = 0. Hence to get a stable economy the initial shock needs to lie on the expectation path that we did not assume to hold in this part. Once this condition is fulfilled, we also know that the logs of EER and relative prices gaps then follow decreasing exponential functions of time with rate of convergence θ_1. From this result[12] we can infer easily that (2) holds continuously.

At this stage it may be tempting to claim that the model endogenously provides the expectation path (2). But this would overlook that the initial condition (28) is all but common. Indeed the price shock must exactly relate to EER gap in a unique way, i.e. linearly with coefficient $-1/\lambda\theta_1$. If it doesn't then the economy is unstable and (2) doesn't hold.

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[12] and from perfect foresight.

From a mathematical standpoint, (28) is weaker than (6), and therefore the model gains in generality by assuming (28) rather than (6) or equivalently (2). But this doesn't make sense from an economic point of view, for (28) is too specific. Indeed, what is the economic rationale for assuming (6) or equivalently (2) only at time t = 0 and not later ? We don't see any, and so probably did Dornbusch when he posits (2) at anytime.

IV.3 About Frankel's approach

Following Frankel we could have described the economy with (23) and (24), that is with a set of variables which are not gaps. The economy can then be represented in the following matrix form:

$$\dot{X}(t) = AX(t) + B(t)U(t) \tag{29}$$

where:

$$X(t) = \begin{bmatrix} e \\ p-p* \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1/\lambda \\ 2\delta\pi & -(\sigma/\lambda+2\delta)\pi \end{bmatrix} \quad B = \begin{bmatrix} \phi/\lambda & -1/\lambda \\ (\gamma-1-\sigma\phi/\lambda)\pi & \sigma\pi/\lambda \end{bmatrix} \quad U(t) = \begin{bmatrix} y-y* \\ m-m* \end{bmatrix}$$

X(t) is the vector of state variables, U(t) the vector of control variables, and A and B are time invariant matrices. U(t) depends only on exogenous variables and deserves to be defined as the control vector.

We have already said that Frankel supposes that the dynamics of the goods market does not depend on the contemporanous interest rate. Therefore he gets different values of A, B and U(t):

$$A = \begin{bmatrix} 0 & 1/\lambda \\ 2\delta\pi & -2\delta\pi \end{bmatrix} \quad B = \begin{bmatrix} -1/\lambda & 1 \\ 0 & 1 \end{bmatrix} \quad U(t) = \begin{bmatrix} \bar{p}-\bar{p}* \\ \Pi-\Pi* \end{bmatrix}$$

In both cases, the state equation is typical of a constrained — or forced — linear continuous-time system. The general solution of (29) is:

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}B(\tau)U(\tau)d\tau$$

$e^{At}X(0)$ is the solution of the free system and is explicited in IV.3.

Hence Frankel's solution is the sum of a vector of the type

$$\begin{bmatrix} a_1e^{-\theta_1 t} + a_2e^{-\theta_2 t} \\ b_1e^{-\theta_1 t} + b_2e^{-\theta_2 t} \end{bmatrix}$$

and of the vector $\int_0^t e^{A(t-\tau)} B(\tau)U(\tau)d\tau$.

$\int e^{A(t-\tau)}B(\tau)U(\tau)d\tau$ can prove to be difficult to obtain and chiefly has no reason to be fully made of terms in $e^{-\theta_1 t}$ and $e^{-\theta_2 t}$. For instance, suppose the long-run economy grows at steady rates, and in particular exhibits steady inflations Π and Π^*. Then $\bar{p}$ and $\bar{p}^*$ are linear functions of time and $\int e^{A(t-\tau)}B(\tau)U(\tau)d\tau$ carries terms in $te^{-\theta_1 t}$ and $te^{-\theta_2 t}$.

Therefore approaching the problem with non-gap variables, as Frankel did, doesn't allow to assess that the endogenous variables — in logs for the exchange rate and prices — converge exponentially to their long-term equilibrium values[13].

Yet there exists one special case where it is allowed: this is when all exogenous variables are held constant. But we then get back to Dornbusch's framework.

V. Conclusion

In this paper we have shown that by using the same model as Dornbusch's but without its restrictive assumptions of no growth in the long-run macroeconomic variables and of a small country, we keep the same kind of results as with the original model and moreover as with Frankel's ''real interest rate model''. Briefly speaking, setting Dornbusch model in motion brings Frankel model.

Unfortunately, we also have proved that the expectations formation process at the heart of both models is unlikely to hold endogenously.

[13] provided the initial condition (28) be met.