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The S Option
- An alternative to the Surrender Option
in Mortgage-backed Securities -

Andrea RONCORONI*

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*Etudiant en thèse au CEREG (Université Paris Dauphine) et ESSEC

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Andrea Roncoroni
Université de Paris Dauphine &
E.S.S.E.C. Graduate Business School, France

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1. Mortgage-backed Securities and the S Option

A mortgage-backed security (MBS) entitles the holder to a unit quota of rights over a pool of mortgages. The contractual provisions affect three orders of counterparts: a financial institution holding a portfolio of residential mortgages, a number of householders and other investors wishing for liquid fixed-income securities. The financial institution decides to sell part of its residential mortgages by putting them into a pool and selling stakes of it to investors. A unit entitles the holder to receive a corresponding quota of both principal and interest cash flows. A market where these units may be traded and a guarantee set up by government-related agencies ensure liquidity for these assets. The financial institution setting up the pool gains cash and the investors own a financial security whose credit rating is comparable to that of regular fixed-income investments, if not for the major feature of an MBS: the householder of a mortgage has always the right to repay the whole principal thus extinguishing his own debt. From a financial point of view, the investors participating to the pool buy a security, the MBS, embedding short positions in american options, one for each underlying mortgage, that may be exercised by the householder at any time up to the mortgage maturity. It is

⁰CEREG, Université de Paris Dauphine, Pl. du Maréchal de Lattre-de-Tassigny, 75775 Paris, France; Tel. +33603910992; Fax. +33144054023; Dept.of Finance, ESSEC Graduate Business School, Av. Bernard Hirsch BP 105, 95021 Cergy Pontoise, France; *e-mail: roncoron@cims.nyu.edu*

arguable that when interest rates decrease, the householder will find convenient to repay the principal and look for entering a new mortgage with a lower rate. This is not very welcome by the investors who see their investment unexpectedly erased. Indeed this optionality feature is taken into account at the moment of buying an MBS by considering a prepayment function describing at each time t the expected rate of prepayments in terms of the time t yield curve.

This contractual scheme may have some inconveniencies for all the involved counterparts: facing the costs of resettlement of a new mortgage, householders may not get profit of small yet potentially valuable decreases in interest rates levels; investors in MBS need to dispense efforts and money to seek for other investments of the same quality; financial institutions may wish to maintain a link with both the householders and the investors. These facts may require a new financial product that replicates the most of the benefits from holding an MBS and nevertheless assures the counterparts against the just mentioned drawbacks..

We hereby propose a new derivative product, the S option, consisting in a set of consecutive american options attached to a *pro-rata* repayment policy for a mortgage.

The contractual provisions may be organized along the following lines:

- at the outlet, a maximal number of refinancements is defined once for all: a refinancement is equivalent to setting up a new mortgage for the remaining life of the previous one, but this is set at a lower market interest rate;
- a stream of periodical payments of interest rate plus principal quotas begins at a starting rate;
- along the repayment period the debtor may refinance his debt by choosing between the rate he already pays off and a rate of reference in the market; this is allowed to as long as the last option has not been exercised yet; if this case occurs, then the debtor will have to end paying his debt according to the last rate ever set up.

From a purely economic point of view, this security provides the debtor with the possibility to take benefit from *any* market interest rate falls by allowing him to refinance the effectively paid off interest, called *reference* interest, due up to the end of the repayment period, this occurs unless a new refinancing, if still allowed, is chosen. Moreover he avoids to incur into entrance fees for getting into a new mortgage each time he quits the one he has been previously entered

in. Finally since he is not allowed to indefinitely refinance, he might ask the financial institution for a negative spread in the interest payments in order to get compensation. For the creditor party, either the financial institution or the investors in the MBS, the S option guarantees the absence of sudden repayments and the consequent need to reallocate the capital elsewhere. The investment cashflows are also more definite and the link with the householder is guaranteed up to default of this latter. From the viewpoint of financial engineering, this contract consists in a finite ordered sequence of american options each one written on a function of the current rate of reference and the value the following option: if one has for instance five options to refinance and at a certain time the first two of them have already been exercised, then it should be decided whether to exercise the third one and get advantage of the current lower rate or to keep on holding all of the three remaining options; evidently this decision will be based upon the comparison between the compounded value of the residual three options and the sum of the value of interest gain plus the compounded value of the last two options. From a mathematical viewpoint, the option valuation problem is a stochastic dynamic programming one, with a discrete time horizon and a mixed continuous-discrete state space of somewhat awkward features. Indeed the main difficulty lays in setting up the problem with a proper state variable. This can be done so to make it equivalent to a completely discrete state space problem, thus providing us with an easily implementable algorithm. The randomness is carried over by the *market* interest rate, whose dynamics is supposed to follow a discrete time-discrete space Markov process. The control policy consists in deciding whether to set up the reference interest rate to the market one or to go on with the previously fixed rate. The mathematics and the nature of the problem are unaffected by adding constant negative spread all over the involved rates, as it might be the case in a more realistic instance. At each date along the repayment period, the debtor must take this decision as long as he has not yet filled up all the number of refinancing opportunities, in the following named *refinance options*. The objective is to provide a mathematical description for this security so to price it and find the optimal refinancing policy.

The corresponding optimal stopping problem is solved by the dynamic programming principle. An algorithmic solution is presented with detailed explanation about some tricks needed to simplify the developed procedure.

References on MBS valuation are [Cheyette, 1992], [Stanton, 1990] and [Hull, 1997]; for an interesting application to life insurance see [Albizzati and Geman, 1994].

2. Mathematical Formulation of the Problem

2.1. Technical Description: Setting Up the Security

Let us consider a finite horizon discrete set of times $\mathcal{T}_0 := \{0, 1, \dots, T\}$ and denote the set $\mathcal{T}_0 \setminus \{0\}$ by \mathcal{T} . Suppose at time 0 an economic agent receives a loan of 1 Euro, generating an outstanding balance $B(0)$ of the same amount to be repaid together with interest all over \mathcal{T}_0 according to a *pro rata* scheme. This means that for each $t \in \mathcal{T}$ an interest, which by now we suppose to be computed by a deterministic annual rate $r(t)$, is calculated on the outstanding balance $B(t-1)$ and then added to it so to come up to the total exposure $B'(t)$ of the debtor at time t . This is virtual because, just at time t , $P(t)$ Euros, calculated as a fraction $1/(T-(t-1))$ (proportional to the number of remaining days to end) of this total exposition, are paid off so that the effective time t outstanding position of the debtor is $B'(t) - P(t) := B(t)$. Once the computation of this interest has been carried out for each t , the outstanding process $\{B(t), 0 \leq t \leq T\}$ and the payment process $\{P(t), 0 < t \leq T\}$ are fully determined. Finally the 'cost to go' for that sequence of payments is defined by the sum of the values of all the payments $\{P(t), 0 < t \leq T\}$. We remark that $B(0) = 1$ as at the outlet no interest has already been accrued, $B(T) = 0$ since $P(T) = 1/(T-(T-1)) B'(T) = B'(T)$, thus $B(T) = B'(T) - P(T) = 0$.

More realistically we suppose that the interest rate $r(t)$ is stochastic and it is determined as follows. Let $\{R(t), 0 \leq t \leq T\}$ be the randomly fluctuating market interest rate, as described by a Markov chain with state space $\mathcal{S} := \{s_{\min} + k\Delta s, k = 0, 1, \dots, K\}$, where the minimum admissible rate s_{\min} , the interest lag Δs and the cardinality $K + 1$ of the set of admissible values for the interest rate are all *a priori* fixed. At time 0, the reference rate $r(0)$ for computing the interest is supposed to be the known value $R(0)$. At a generic time t , the debtor is allowed to either set the reference rate $r(t)$ equal to the current market rate $R(t)$ or to leave it at the level it was one time lag before, namely at $r(t-1)$. We impose this option to be exercisable only up to a fixed number of times N along with the repayment period; in order to prevent our contract from triviality, let $N < T$: if the equality holds, the optimal strategy would be to keep on changing the rate each time the market one is lower than the previous reference rate and this could be done as often as one wishes.

A synthesis of the problem together with further data such as the transition probabilities for the Markov chain are hereby outlined.

2.2. Model Data

- $\mathcal{T}_0 := \{0, 1, \dots, T\}$, $T = 250$ weeks is the time horizon
- $\mathcal{T}_0 \setminus \{0\} := \mathcal{T}$
- $\mathcal{S} := \{s_{\min} + k\Delta s, k = 0, 1, \dots, K\}$ is the space of admissible rates, where $\Delta s = 0.002$ and $K = 45$
- $R(0) := 0.05$ is the initial market rate
- $\{R(t), t \in \mathcal{T}_0\} : \forall x \in \mathcal{S}, p(x, \cdot)$ is the transition probability function for the process $R(t)$ representing the market interest rate:

$$P(x, y) = \begin{cases} P_N(x, y) = \begin{cases} \frac{1}{3} & \text{if } y = x + 0.002 \\ \frac{1}{3} & \text{if } y = x \\ \frac{1}{3} & \text{if } y = x - 0.002 \\ 0 & \text{elsewhere} \end{cases}, & 0.01 < x < 0.1 \\ P_-(x, y) = \begin{cases} \frac{1}{2} & \text{if } y = 0.01 (= s_{\min}) \\ \frac{1}{2} & \text{if } y = 0.012 (= s_{\min} + \Delta s) \end{cases}, & x = 0.01 = s_{\min} \\ P_+(x, y) = \begin{cases} \frac{1}{2} & \text{if } y = 0.1 (= s_{\max}) \\ \frac{1}{2} & \text{if } y = 0.098 (= s_{\max} - \Delta s) \end{cases}, & x = 0.1 = s_{\max} \end{cases} \quad (2.1)$$

inducing a probability measure P_x on the space of paths if the initial condition is x

- $n(0) = 4$ is the number of refinancing opportunities available at the outlet ($t = 0$) of the loan repayment period, then it is a controlled stochastic process $\{n^\alpha(t)\}$, where $\alpha(t+1) : \text{state variable } \mathbf{x} \rightarrow D(\alpha(t))$ is the time $t+1$ control determined by the status of the state variable corresponding to that time and taking values in $D(\alpha(t)) := \{0, 1\}$ if $\sum_{i=1}^t \alpha(i) < 4$, $\{0\}$ otherwise (if no more than 3 refinancements have been done, than a further one is allowed to occur): 0 stands for 'not refinance', 1 for 'refinance'; the state variable is the set of quantities whose knowledge lets one choose a control; we will take the task of its determination in the next paragraph.
- $r(0) := R(0)$ is the initial reference rate which we set to the value of the market one
- $r^{\alpha(t)}(t)$ is the time t reference interest rate; it is set to $R(t)$ if $\alpha(t) = 1$ or to $r^{\alpha(t-1)}(t-1)$ if $\alpha(t) = 0$

- $B(0) := 1$ is the initial outstanding equal to the borrowed capital
- $B(t+1)$ is the time $t+1$ outstanding capital recursively defined by:

$$\begin{aligned}
B(t) &\rightarrow B'(t) := B(t) \left(1 + \frac{r^{\alpha(t)}(t)}{52} \right) \\
&\rightarrow P(t) := \frac{B'(t)}{T-t} \rightarrow B(t+1) := B'(t) - P(t)
\end{aligned}$$

2.3. The Objective Functional

$$\min_{\alpha \in \mathcal{A}} E^{P_{r_0}} \left(\sum_{j=1}^T P^\alpha(j) \right)$$

where \mathcal{A} is the class of control policies $\{\alpha_1, \dots, \alpha_T\}$ determining the process $\{P^\alpha(t), 0 \leq t \leq T\}$ on $(\mathcal{S}^T, \mathcal{B}(\mathcal{S}^T), P_{R(0)})$ and $E^{P_x}(\cdot)$ is the expectation operator with respect to the probability measure induced by the transition probabilities above stated and the initial condition x , over the space of paths.

The next representation will be useful for shortening computations:

$$\begin{aligned}
B(t+1) &= B'(t) - P(t) \\
&= B'(t) - \frac{B'(t)}{T-t} \\
&= \left(1 + \frac{r^{\alpha(t)}(t)}{52} \right) B(t) \left(1 - \frac{1}{T-t} \right) \\
&\triangleq c(r^{\alpha(t)}(t)) B(t) \tau(t)
\end{aligned}$$

where: $c(x)$ is the accruing factor corresponding to an interest rate x and $\tau(x)$ is the residue coefficient corresponding to time x .

3. Recursive Valuation

3.1. The Procedure for State Variable Identification

The state variable of a dynamic programming problem is the set of quantities upon which the control is chosen. At time t they all must be available in order to uniquely determine the control for that time.

In order to guess the right variable, it is useful to recall the procedure above described. The tern $(B(t), r^{\alpha(t)}, t)$, representing time t outstanding balance, the previously set up reference rate and the date t itself, fully determines the pair $(B'(t), P(t))$, namely the time t updated capital-plus-interest outstanding and *pro rata* payment at that time, and $P(t)$ goes into the objective functional. When $\alpha(t)$ is looked for, the information $(B(t), t)$ is available and since, as just seen, this couple affects the objective functional, one might argue that $\alpha(t)$ depends on $(B(t), t)$. What is the effect of $\alpha(t)$? Actually $\alpha(t)$ states whether the new reference rate $r(t)$ is to be updated to $R(t)$ or to remain equal to its previous value $r(t-1)$. Taking this decision amounts to compare the effects of both of the choices and these outcomes depend on the values of $R(t)$ and $r(t-1)$: $\alpha(t)$ is a function of $R(t)$ and $r(t-1)$. Finally, one must not forget that the option of choosing among $R(t)$ and $r(t-1)$ exists until $n \leq 4$: afterwards it expires. Conclusion: the time t control $\alpha(t)$ is, in principle, a function of t , the time t outstanding amount $B(t)$, the time t market interest rate $R(t)$, the reference rate $r(t-1)$ in charge one time before and the number $n(t)$ of still available options to refinance.

$$\alpha(t)(t, B(t), R(t), r(t), n(t)) = F(s, y, x, r, n) |_{(t, B(t), R(t), r(t), n(t))}$$

The 5-uple (s, y, x, r, n) is our *candidate* for a state variable: in what follows we indicate it as $(s, (y, x, r, n))$ to underlie its meaning of *time-state variable*.

This is a \mathcal{F}_t -measurable function, thus it is admissible as a control policy.

The issue is to find F . We will show that while F is truly dependent on s, y, x, r, n , that is the value of $t, R(t), r(t-1)$ and $n(t)$.

3.2. State Variable Domain Restrictions

We are going to recursively define the value function relative to the studied problem and select the optimal refinance strategy consisting in a 4-uple of \mathcal{T} -valued strictly increasingly ordered stopping times.

One element of the state variable, namely $B(t)$, takes value in \mathbb{R}^+ and this could in principle prevent us to write down a finite list of possible outcomes for the value function in correspondence of all the values of the state variable at a given time. The candidate time-state variable $(t, (y, x, r, n))$ is defined on a subset of the non discrete set:

$$\mathcal{X}^{t, B, R, r, n} := \mathcal{T} \times (\mathbb{R}^+ \times \mathcal{S} \times \mathcal{S} \times \{0, 1, 2, 3, 4\})$$

We say 'subset' because natural restrictions apply. For example, if $t = 1, 2, 3, 4$, then n takes value in respectively, $\{4\}$, $\{3, 4\}$, $\{2, 3, 4\}$ and $\{1, 2, 3, 4\}$; at time 1 all the refinance options are still available since at 0 no choice has been performed; if $t = 1$, then r must be equal to r_0 ; further, as $t = 2$, $r \in \{r_0, r_0 + \Delta s, r_0 - \Delta s\}$. Since our purpose is to solve a problem by an easily implementable algorithm, the saving from a smaller number of computations for the value function will be overwhelmed by the increase of complexity due to the definition of the precise state variable domain for the value function at each time, so that we do not take care of these constraints along the implementation procedure.

Continuing on the point we are making, we will show that *the value function is homogeneous of degree 1 in y* . Thus it can be factored as:

$$V(t, (y, x, r, n)) := y \cdot V^*(t, (x, r, n))$$

We refer to V^* as the *unitary value function* since it is the value function per unity of outstanding balance.

Recall that y is nonnegative and that V is a minimum function among two numbers. Therefore the smallest, which determines V , is selected irrespective to y and thus the optimal policy determining V is the one establishing V^* : the optimal stopping rule is independent of y and has form:

$$F = F(x, r, n) \tag{3.1}$$

We remark the independence of t .

From now on F will denote the functional defined by (3.1), whose domain is a subset of:

$$\mathcal{X} := \mathcal{X}^{R,r,n} := \mathcal{S} \times \mathcal{S} \times \{0, 1, 2, 3, 4\}$$

We proceed by steps, firstly by computing the value function for times $T, T - 1, T - 2$, then for a general t and finally for $t = 1$ and 0, the aim being the calculation of the latter.

3.3. Algorithm

3.3.1. Time T Value Function

$$V(T, (y, x, r, n)) = 0$$

because at time T the debtor pays off the time $T - 1$ outstanding capital plus the interest accrued between $T - 1$ and T and the contract extinguishes, so that there are no more future cash flows.

3.3.2. Time T-1 Value Function

If $n = 0$, *id est* no refinance option is available anymore, then the debtor incurs into a payment equal to time $T - 1$ outstanding y plus the interest accrued on it between $T - 1$ and T according to the rate r , namely:

$$CNR^0(T - 1, y, x, r) = y \cdot c(r) \cdot \tau^*(T - 1)$$

where: $c(r) := (1 + \frac{r}{52})$, $\tau^*(T - 1) := (1 - \frac{1}{T-t})|_{t=T-1}$, $\tau^*(t) := \frac{1}{T-t}$ is the *pro-rata* quota and CNR^0 is the acronym for 'Cost of Not Refinance in the case of no, namely 0, available refinance options.

If $n > 0$, then we have to minimize the Cost associated to the two available policies of Refinancing, CR , and Not Refinancing, CNR computed as follows.

$CR(T - 1, y, x, r)$ is the sum of the Cost for Period $[T - 1, T]$ due to interest then accrued on y according to rate x , denoted by $CP_{[T-1,T]}(y, x)$, plus the Discounted Expected future optimal Cost starting from $(T - 1) + 1$, denoted by $DEC(T, y, x, r, n)$:

$$\begin{aligned} CR(T - 1, y, x, r, n) &= CP_{[T-1,T]}(y, x) + DEC(T, y, x, r, n) \\ &= y \cdot c(x) \cdot \tau^*(T - 1) + 0 \end{aligned}$$

$CNR(T - 1, y, x, r, n)$ is the same except for the rate which is set to be equal to r since no refinance option has been exercised:

$$\begin{aligned} CNR(T - 1, y, x, r, n) &= CP_{[T-1,T]}(y, r) + DEC(T, y, x, r, n) \\ &= y \cdot c(r) \cdot \tau^*(T - 1) + 0 \end{aligned}$$

The value function is:

$$\begin{aligned} V(T - 1, (y, x, r, n)) &= \begin{cases} \min \{CR(T - 1, y, x, r, n), CNR(T - 1, y, x, r, n)\} \\ \text{if } n = 1, 2, 3, 4 \end{cases} \\ &= \begin{cases} CNR^0(T - 1, y, x, r) \text{ if } n = 0 \\ \min \{y \cdot c(x) \cdot \tau^*(T - 1), y \cdot c(r) \cdot \tau^*(T - 1)\} \\ \text{if } n = 1, 2, 3, 4 \\ y \cdot c(r) \cdot \tau^*(T - 1) \text{ if } n = 0 \end{cases} \end{aligned} \quad (3.2)$$

If $n > 0$, the optimal decision is to refinance if and only if the market rate x is less than the reference rate r . This is because $c(\cdot)$ is monotonically increasing a function and both y and $\tau^*(T - 1)$ are positive.

Conclusion: the optimal stopping rule for time $T - 1$ is

$$\alpha(T - 1)(y, x, r, n) = \alpha(x, r, n) = \begin{cases} 1 \text{ (refinance)} & \text{if } n > 0 \text{ and } x < r \\ 0 \text{ (not refinance)} & \text{if } n = 0 \text{ or } r \leq x \end{cases} \quad (3.3)$$

While the value function depends on the entire time-state variable, the optimal decision rule does not: it is a function of only $(t, (x, r, n))$, referred to as the *true time-state variable*.

3.3.3. Time T-2 Value Function

Time $T - 2$ value function is to be computed as in (3.2) where $T - 1$ is replaced with $T - 2$:

$$V(T - 2, (y, x, r, n)) = \begin{cases} \min \{CR(T - 2, y, x, r, n), CNR(T - 2, y, x, r, n)\} \\ \text{if } n = 1, 2, 3, 4 \\ CNR^0(T - 2, y, x, r) \text{ if } n = 0 \end{cases} \quad (3.4)$$

The main difference lies in longer computation needed to find out the involved quantities.

$$\begin{aligned} CR(T - 2, y, x, r, n) &= CP_{[T-2, T-1]}(y, x) + DEC(T - 1, y, x, r, n) \\ &= y \cdot c(x) \cdot \tau^*(T - 2) \\ &\quad + E^{P_x}(V(T - 1, (B(T - 1), \cdot, x, n - 1))) \end{aligned} \quad (3.5)$$

where \cdot indicates the argument with respect to which expectation is computed.

If the debtor decides to refinance, then an interest on the time $T - 2$ outstanding y accrues from $T - 1$ to $T - 2$ according to the time $T - 2$ market rate x ; the mechanism of *pro-rata* reimbursement previously described give rise to a payment for the period between $T - 2$ and $T - 1$ equal to $y \cdot c(x) \cdot \tau^*(T - 2)$; to this, one must add the best (*id est* lowest) expected payment from $T - 1$ to the end of the contract life; as for its absolute value, given each possible realization $R(T - 1)(\omega)$ of the random variable $R(T - 1)(\cdot)$, one has a best residual total cost equal to the time $T - 1$ value function $V(T - 1, (y^*, x^*, r^*, n^*))$ computed at:

- an outstanding balance equal to:

$$B(T - 1) = c(r(T - 1)) \cdot B(T - 2) \cdot \tau(T - 2)$$

because time $T - 1$ reference rate is equal to the rate chosen at $T - 2$ by definition and here this is time $T - 2$ market rate, that is x ;

- time $T - 1$ market interest rate equal to $R(T - 1)(\omega)$;
 - time $T - 1$ reference rate equal to the rate used on $[T - 2, T - 1]$, namely x ,
- and
- the number of still available refinance options equal to the ones held at $T - 2$ minus the one spent there, that is $n - 1$.

Thus V must be computed at point $(T - 1, (c(x) y \tau(T - 2), R(T - 1), x, n - 1))$; this is the random variable over which we average by taking expectation with respect to the transition probabilities for $R(T - 2)$ starting at x : this explains the notation E^{P_x} in formula (3.5).

Factorizing V as $y \cdot V^*$ and plugging into (3.5) one gets:

$$\begin{aligned}
CR(T - 2, y, x, r, n) &= yc(x) \tau^*(T - 2) \\
&\quad + E^{P_x}(V(T - 1, (c(x) y \tau(T - 2), R(T - 1)(\cdot), x, n - 1))) \\
&= yc(x) \tau^*(T - 2) \\
&\quad + E^{P_x}(c(x) y \tau(T - 2) V^*(T - 1, (R(T - 1)(\cdot), x, n - 1))) \\
&= y(c(x) \tau^*(T - 2) \\
&\quad + c(x) \tau(T - 2) E^{P_x}(V^*(T - 1, (R(T - 1)(\cdot), x, n - 1))))
\end{aligned} \tag{3.6}$$

As for CNR analogous computations lead to:

$$\begin{aligned}
CNR(T - 2, y, x, r, n) &= CP_{[T-2, T-1]}(y, r) + DEC(T, y, x, r, n) \\
&= y \cdot c(r) \cdot \tau^*(T - 2) \\
&\quad + E^{P_x}(V(T - 1, (B(T - 1), \cdot, r, n)))
\end{aligned} \tag{3.7}$$

where r is used as reference rate over $[T - 2, T - 1]$, so that no refinance has been carried out; therefore time $V(T - 1, (\cdot, \cdot, \cdot, \cdot))$ is to be computed for a time $T - 1$ outstanding balance equal to $c(r) \cdot y \cdot \tau(T - 2)$, time $T - 1$ market rate $R(T - 1)(\omega)$, reference rate r and unchanged number of refinance opportunities, that is n . The factorization of V into $y \cdot V^*$ gives:

$$\begin{aligned}
CNR(T - 2, y, x, r, n) &= y(c(r) \tau^*(T - 2) \\
&\quad + c(r) \tau(T - 2) E^{P_x}(V^*(T - 1, (R(T - 1)(\cdot), r, n))))
\end{aligned} \tag{3.8}$$

Finally $CNR^0(T - 2, y, x, r)$ is nothing but the cost of not refinancing whenever this is the only viable options: this is the case of $n = 0$. Therefore one gets:

$$CNR^0(T - 2, y, x, r) = CNR(T - 2, y, x, r, 0)$$

Combining formulas (3.6), (3.7) and (3.8), we come up to a computable expression for time $T - 2$ value function:

$$V(T-2, (y, x, r, n)) = \begin{cases} \min \{ y [c(x) \tau^*(T-2) + c(x) \tau(T-2) E^{P_x}(V^*(T-1, (R(T-1)(\cdot), x, n-1)))] , \\ y \cdot c(r) \cdot \tau^*(T-2) + E^{P_x}(V(T-1, (B(T-1), \cdot, r, n))) \} \\ \text{if } n = 1, 2, 3, 4 \\ y (c(r) \tau^*(T-2) + c(r) \tau(T-2) E^{P_x}(V^*(T-1, (R(T-1)(\cdot), r, 0)))) \\ \text{if } n = 0 \end{cases}$$

Again the practical problem of storing data into the computer can be solved by looking at the factorized corresponding problem. For each $(x, r, n) \in \mathcal{S} \times \mathcal{S} \times \{0, 1, 2, 3, 4\}$ store the numbers:

$$V^*(T-2, (x, r, n)) = \begin{cases} \min \{ [c(x) \tau^*(T-2) + c(x) \tau(T-2) E^{P_x}(V^*(T-1, (R(T-1)(\cdot), x, n-1)))] , \\ c(r) \cdot \tau^*(T-2) + E^{P_x}(V^*(T-1, (B(T-1), \cdot, r, n))) \} \\ \text{if } n = 1, 2, 3, 4 \\ (c(r) \tau^*(T-2) + c(r) \tau(T-2) E^{P_x}(V^*(T-1, (R(T-1)(\cdot), r, 0)))) \\ \text{if } n = 0 \end{cases}$$

The optimal stopping rule for time $T - 2$ is no more as simple as in (3.3). An explicit computation of the above minimization must be carried out in order to choose whether or not refinancing. We remark that the probability under which expectation are taken is determined by (2.1) through: $P_x(\{y\}) := P(x, y)$ and it is the same in both of the cases since the driving random process is $R(t)$ starting at $R(T-2) = x$.

3.3.4. Time t Value Function

In general, for $0 \leq t \leq T$, we have:

$$V^*(t, (x, r, n)) = \begin{cases} \min \{ c(x) \tau^*(t) + c(x) \tau(t) E^{P_x}(V^*(t+1, (R(t+1)(\cdot), x, n-1))) , \\ c(r) \cdot \tau^*(t) + E^{P_x}(V^*(t+1, (B(t+1), \cdot, r, n))) \} \\ \text{if } n > 0 \\ (c(r) \tau^*(t) + c(r) \tau(t) E^{P_x}(V^*(t+1, (R(t+1)(\cdot), r, 0)))) \\ \text{if } n = 0 \end{cases}$$

3.3.5. Time 1 Value Function

At time 1 the unitary value function V^* degenerates with respect to the variables r and n : in fact this is the first time the debtor is allowed to refinance, thus he or she has still all of the refinance options set up at outlet, that is 4; also, the rate used on $[0, 1]$ is r_0 ; finally $R(1) \in \{r_0, r_0 + \Delta s, r_0 - \Delta s\}$.

Therefore one has to compute $V^*(1, (x, r, n))$ only for $x \in \{r_0, r_0 + \Delta s, r_0 - \Delta s\}$, $r = r_0$, $n = 4$.

Such restrictions start arising at $\max\{r_0 - s_{\min}, s_{\max} - r_0\} / \Delta s$, that is the first time at which the random variable $R(t)$ can touch the farthest barrier from the starting point r_0 ; then r can not be over that barrier, so that V^* is no worth to be computed at r equal to that barrier. Going backward, the number of needed values for r gets smaller and smaller, up to time 1 as it collapses to one, that is $r = r_0$.

A similar reasoning applies to the variable n : at time 4, n can not be 0; at time 3, it is not allowed to be neither 0, nor 1; at time 2 the only admissible values are $n = 3, 4$; the last case has been just examined.

As already mentioned, this set of natural restrictions are of no use for practical purposes: the cost of implementation exceed the decreasing of efficiency in terms of time loss due to inessential computations.

We decide to compute V^* over $\mathcal{X}^{t,R,r,n} := \mathcal{T} \times (\mathcal{S} \times \mathcal{S} \times \{0, 1, 2, 3, 4\})$.

3.3.6. Time 0 Value Function: the Price of the Contract

At time 0, $x = r_0$, r no more exists, so that V^* is independent of the variable r , $n = 4$ and $V = V^*$ since $y = 1$. One has:

$$\begin{aligned} V(0, (y, x, n)) \Big|_{(1, r_0, 4)} &= V^*(0, (1, r_0, 4)) \\ &= c(r_0) 1\tau^*(0) + c(r_0) 1\tau(0) E^{P_{r_0}}(V^*(1, (\cdot, r_0, 4))) \end{aligned}$$

which is the price of the contract at time 0.

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